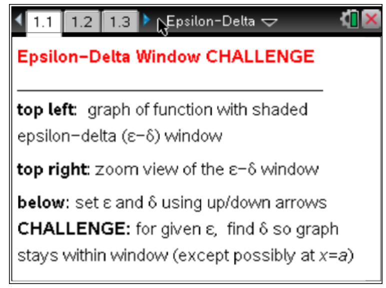

Math Objectives

- Students will interpret the formal definition (epsilon-delta) of *limit* in terms of graphing window dimensions.
- Students will use this interpretation to make judgments of whether a limit exists for particular examples.
- Students will construct viable arguments and critique the reasoning of others. (CCSS Mathematical Practice)
- Students will use appropriate tools strategically. (CCSS Mathematical Practice)

Vocabulary

- limit of a function

About the Lesson

- The intent of the lesson is to help students make sense out of the formal mathematical definition of *limit.* This does *not* mean that students will be expected to be proficient at writing "epsilon-delta" proofs as a result of this activity, but the activity could serve as an introduction to further work that includes proofs.
- The requirements of the formal definition of limit are put in terms of a graphing window challenge game between two opposing players. Students are given the question: is $\lim_{x \to a} \mathbf{f}(x) = L$?

The game proceeds as follows: a graphing window must be set with (a, L) in the center of the screen.

1. The student's opponent always goes first and sets YMin = $L - \varepsilon$ and YMax = $L + \varepsilon$.
2. The student goes next and sets XMin = $a - \delta$ and XMax = $a + \delta$.

The student wins the challenge if the graph stays entirely within the window from left to right, except possibly at $x = a$.

- There are a series of examples posed in this graphing window challenge format. The examples are chosen to display important kinds of continuous and discontinuous function behavior, including removable discontinuities, jump discontinuities, and even damped and undamped "wild" oscillatory behavior (in the last two examples).
- Note: The controls for epsilon (up/down arrows on the left) and delta (up/down arrows on the right) have been adapted to switch from arithmetic to exponential scales.

TI-Nspire™ Navigator™ System

- Use Screen Capture and Quick Poll to assess students' understanding as they respond to the questions posed on the

TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Utilize a slider

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- Press ctrl G to either hide the function line or access the function line in a *Graphs* page.

Lesson Materials:

Student Activity

- Epsilon-Delta_Student.pdf
- Epsilon-Delta_Student.doc

TI-Nspire document

- Epsilon-Delta.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

student activity worksheet.

Discussion Points and Possible Answers

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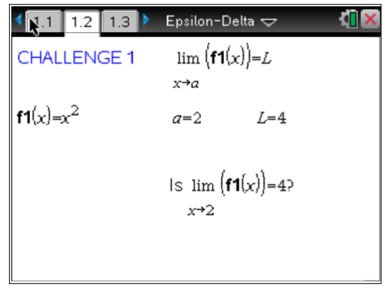


1. If **f1**(x) = x^2, a = 2, and L = 4, then you are interested in the question: is $\lim\limits_{x \to 2} \mathbf{f1}(x) = 4$?

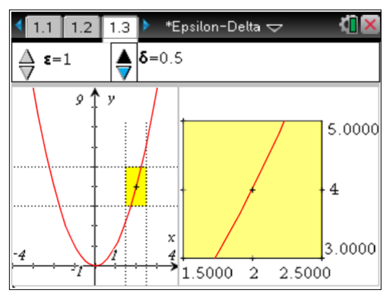
   a. At what point should the epsilon-delta (ε-δ) window be centered?

   <u>Answer:</u> (2, 4) should be the center of the window.

   b. If your opponent sets ε = 1 and you set δ = 0.5, what will be the dimensions (XMin, XMax, YMin, YMax) for the epsilon-delta window?

   <u>Answer:</u> XMin = 1.5, XMax = 2.5, YMin = 3, YMax = 5

Move to page 1.3.



   c. Your opponent chose ε = 1, so set ε = 1 using the up/down arrows on the left. Now use the up/down arrows on the right to set δ = 0.5. Does this value δ = 0.5 meet the challenge? Why or why not? If not, can you find a value δ that does meet the challenge for ε = 1?

   <u>Answer:</u> No. The graph does not enter from the left and exit from the right. δ = 0.2 (or smaller) is an example of a value meeting the challenge.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 1 at the end of this lesson.

   d. Suppose your opponent chooses ε = 0.1. Set this value ε for and use the up/down arrows on the right to find a value for δ that meets the challenge.

   <u>Answer:</u> δ = 0.01 (or smaller) is an example of a value meeting the challenge.

e. Would a value of δ larger than the value you chose meet the $\varepsilon = 0.1$ challenge? Explain your answer.

<u>Answer:</u> This depends on student's answer to part 1d; values greater than $\delta = 0.025$ will not be appropriate.

f. Would a value of δ smaller than the value you chose meet the $\varepsilon = 0.1$ challenge? Explain your answer.

<u>Answer:</u> If student's answer to part 1d is appropriate, then any smaller value would also meet the challenge. This is an important point: if there exists one appropriate δ-value, then any smaller positive δ is also correct.

g. Suppose your opponent chooses $\varepsilon = 0.01$. Set this value for ε and use the up/down arrows on the right to find a value for δ that meets the challenge.

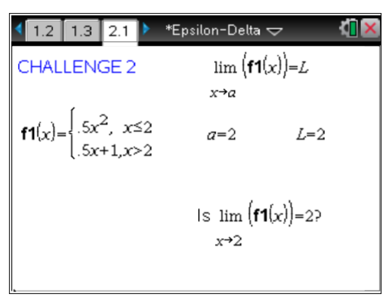
<u>Answer:</u> $\delta = 0.001$ (or smaller) is an example of a value meeting the challenge.

h. Do you believe that you could always win the challenge for any given $\varepsilon > 0$? Why or why not?

<u>Answer:</u> The challenge can always be met. (If epsilon is between 0 and 1, choosing delta to be smaller than $\frac{1}{4}$ of ε will meet the challenge.)

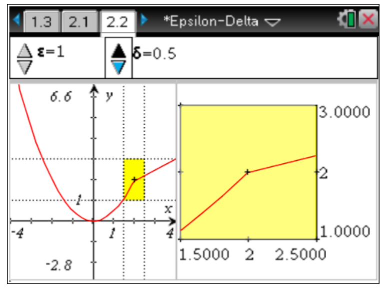
> **Teacher Tip:** The controls on epsilon (up/down arrows on the left) and delta (up/down arrows on the right) do not allow for a setting less than 0.0001, so take care to discuss that technological limitation is not in place for part 1h.

Move to page 2.1.



2. The function **f1** has a split definition, with the formula $\mathbf{f1}(x) = \begin{cases} 0.5x^2, & x \le 2 \\ 0.5x+1, & x > 2 \end{cases}$. You are interested in the question: is $\lim_{x\to 2} \mathbf{f1}(x) = 2$?

Move to page 2.2.



a. Suppose your opponent chooses $\varepsilon = 1$. Set $\varepsilon = 1$ using the up/down arrows on the left. Now use the up/down arrows on the right to set $\delta = 0.5$. Does this value $\delta = 0.5$ meet the challenge? How do you know?

<u>Answer:</u> Yes. The graph enters from the left and exits to the right without leaving the top or bottom of the window.

b. Suppose your opponent chooses $\varepsilon = 0.001$. Set this value for ε and use the up/down arrows on the right to find a value for δ that meets the challenge.

<u>Answer:</u> $\delta = .0005$ (or smaller) is appropriate.

c. Would a value of δ larger than the value you chose meet the $\varepsilon = 0.001$ challenge? Explain your answer.

<u>Answer:</u> This depends on the answer given in part 2b. $\delta = .0005$ is the maximum appropriate value.

d. Would a value of δ smaller than the value you chose meet the $\varepsilon = 0.001$ challenge? Explain your answer.

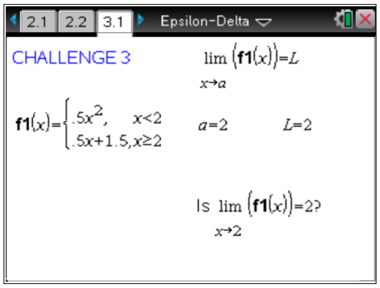
<u>Answer:</u> If the answer in part 2b is appropriate, then any smaller positive value of delta will also be appropriate to meet the challenge.

e. Do you believe that you could always win the challenge for any given $\varepsilon > 0$? Why or why not?

<u>Answer</u>: It is always possible to meet the challenge. (Given any positive epsilon less than 1, setting delta to $\frac{1}{2}$ of epsilon or less will meet the challenge.)

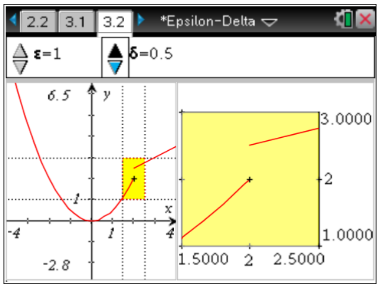
TI-Nspire Navigator Opportunity: ***Quick Poll***
See Note 2 at the end of this lesson.

Move to page 3.1.



3. The function **f1** here also has a split definition, with $\mathbf{f1}(x)=\begin{cases}0.5x^2, & \text{if } x<2\\ 0.5x+1.5, & \text{if } x\geq 2\end{cases}$. You are interested in the question: is $\lim_{x\to 2}\mathbf{f1}(x)=2$?

Move to page 3.2.



a. Suppose your opponent chooses ε = 1. Does the value δ = 0.5 meet the challenge? How do you know?

__**Answer:**__ Yes. The graph enters from the left and exits to the right without leaving the top or bottom of the window. (There is a break at x = 2.)

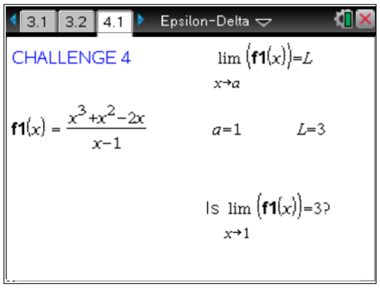
b. Suppose your opponent chooses ε = 0.1. Can you meet this challenge? Why or why not?

__**Answer:**__ The challenge cannot be met! For this epsilon, the graph jumps out through the top of the window immediately to the right of x = 2, so no positive value of delta will meet the challenge.

c. Explain why $\lim_{x\to 2}\mathbf{f1}(x)\neq 2.$ Is there another value L such that $\lim_{x\to 2}\mathbf{f1}(x)=L$ for this function?

__**Answer:**__ Unless you can meet the challenge for every positive epsilon, you cannot say that the limit is 2. You failed in part 3b for ε = 0.1. There is no other value of L that will work. In other words, the limit as x approaches 2 does not exist at all.

Move to page 4.1.

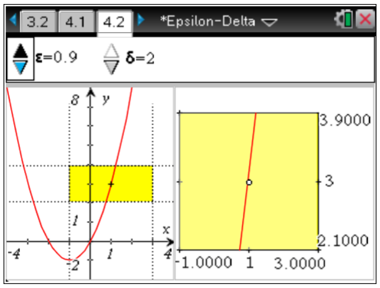


4. The function **f1** is not defined at x = 1.

a. Why is **f1**(1) undefined? Does this mean $\lim_{x\to 1}\mathbf{f1}(x)$ cannot exist?

__**Answer:**__ At x = 1, the denominator is equal to 0. Since division by 0 is not allowed, the function value **f1**(1) is undefined. This does *not* mean that $\lim_{x\to 1}\mathbf{f1}(x)$ does not exist. The value (or lack of any defined value) at x = 1 is irrelevant for the purposes of deciding whether $\lim_{x\to 1}\mathbf{f1}(x)$ exists.

Move to page 4.2.



b. Investigate whether or not $\lim\limits_{x \to 1} \mathbf{f1}(x) = 3$. Try meeting the ε challenge for ε = 0.1, 0.01, and 0.001 by finding an appropriate δ for each.

Answer:

The approximate maximum allowable delta is given (so any smaller delta is also acceptable).

$\varepsilon = 0.1$, $\delta = 0.025$

$\varepsilon = 0.01$, $\delta = 0.0025$

$\varepsilon = 0.001$, $\delta = 0.00025$

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 3 at the end of this lesson.

c. Is it possible to meet the ε challenge for any positive ε by finding an appropriate δ?

Answer: Yes. This would mean that the limit is indeed 3.

Wrap Up

The formal limit definition is perhaps the most difficult encountered in first-year calculus, and it is not suggested here that students must be able to produce polished epsilon-delta proofs before proceeding to calculus concepts that depend on limits (derivatives and integrals, for example). Two important aspects of the definition for students to distinguish are the *quantifiers* for ε and δ:

1) To show $\lim\limits_{x \to a} f(x) = L$ means that for every individual possible positive ε, one can produce a corresponding positive δ that meets the challenge of forcing $L - \varepsilon < f(x) < L + \varepsilon$ for all values x such that $a - \delta < x < a + \delta$ and $x \neq a$.

2) On the other hand, to show $\lim\limits_{x \to a} f(x) \neq L$, one only needs to demonstrate a single positive ε for which it is not possible to produce a positive δ meeting that challenge.

This lesson could be followed up with more work on epsilon-delta proofs of limits. The language of the graphing window challenge game can still be helpful in terms of describing a winning strategy for the second player: given any epsilon value, what's a strategy for always choosing an appropriate delta value that will win that challenge? In practice, for many functions this is an algebraic exercise involving expressing the inequality $L - \varepsilon < f(x) < L + \varepsilon$ as an equivalent inequality of the form $a - \delta < x < a + \delta$ where δ is an expression in terms of ε.

Problems 5, 6, 7, and 8 in the TI-Nspire document provide additional opportunities for investigation or for assessment of student understanding. Comments on these additional problems are provided below.

TI-Nspire Navigator

Note 1

Question 1, *Screen Capture*

This is a good time to take a Screen Capture to ensure students are able to use the sliders to meet the challenge.

Note 2

Question 2, *Quick Poll*

You may want to send a Quick Poll to assess students' understanding. It is important to emphasize that the value of the function at $x = a$ is irrelevant in considering whether or not a function has a limit there.

Suppose **g1** is a function identical to **f1** except that it has a different value at $x = 2$: with the formula $\mathbf{g1}(x) = \begin{cases} 0.5x^2, & x < 2 \\ 3, & x = 2 \\ 0.5x + 1, & x > 2 \end{cases}$

Does $\lim\limits_{x \to 2} \mathbf{g1}(x) = L$ exist? If so, what is L?

Answer: The limit exists and $L = 2$.

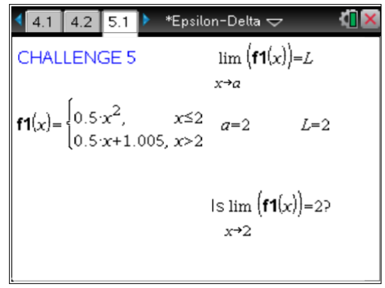
Note 3

Questions 4a and 4b, *Screen Capture*

This is a good time to use Screen Capture to ensure students are following along, and share results with the class.

Extension

Move to page 5.1.

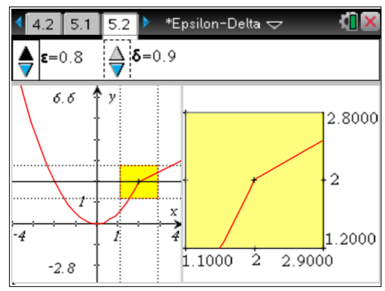


5. On page 5.1, the function **f1** has a split definition. On page 5.2, you will see that the graph looks very much like the function in problem 2.

 a. Is $\lim_{x\to 2} \mathbf{f1}(x) = 2$?

 <u>Answer:</u> No. Zooming in shows that there is a jump discontinuity at $x = 2$.

Move to page 5.2.



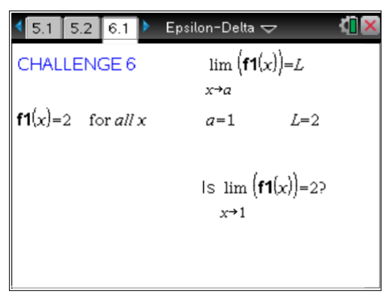
 b. Suppose you get to go first and choose ε. Is there a positive value ε such that there is no δ that meets the challenge?

 <u>Answer:</u> Yes. Choose ε = .0001 and there is no possible delta meeting the challenge.

 c. What does your answer to part 5b say about whether $\lim_{x\to 2} \mathbf{f1}(x) = 2$?

 <u>Answer:</u> Because the challenge cannot be met for every positive ε, the limit cannot be 2.

Move to page 6.1.



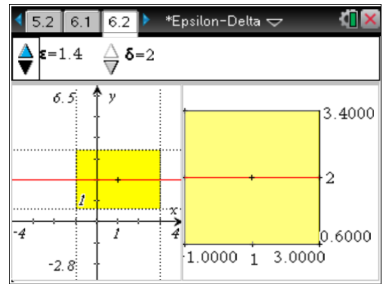
6. Examine the function given on page 6.1.

 a. Is $\lim_{x\to 1} \mathbf{f1}(x) = 2$? Why or why not?

 <u>Answer:</u> Yes. It is easy to find a delta for any given epsilon.

> **Teacher Tip:** As simple as they are, constant functions are sometime problematic for students. (Perhaps it is because the dependent variable does not vary and does not "depend" on the independent variable.)

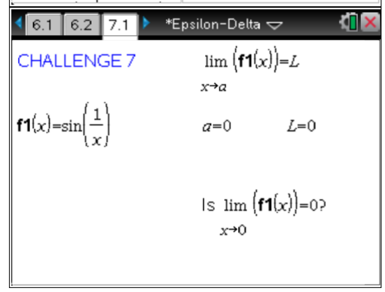
Move to page 6.2.



b. Suppose your opponent sets *ε* = 0.00000000001. What is the *largest δ* you could use to meet the challenge?

Answer: Any *δ*, no matter how big, meets the challenge!

Move to page 7.1.

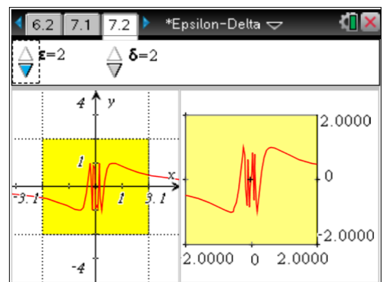


7. $\mathbf{f1}(x) = \sin\left(\frac{1}{x}\right)$

a. This function is undefined at *x* = 0. Why?

Answer: Division by 0 is not allowed.

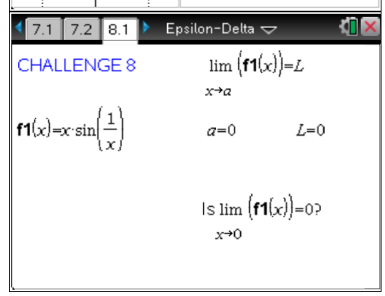
Move to page 7.2.



b. Is $\lim_{x\to 0}\mathbf{f1}(x) = 0$? Defend your answer.

Answer: The limit is not 0. No *δ* can be found to meet the *ε* = 0.5 challenge, for example.

Move to page 8.1.

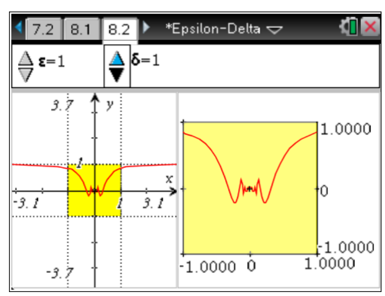


8. $\mathbf{f1}(x) = x \cdot \sin\left(\frac{1}{x}\right)$

a. This function is undefined at *x* = 0. Why?

Answer: Division by 0 is not allowed.

Move to page 8.2.



b. Is $\lim_{x\to 0}\mathbf{f1}(x) = 0$? Defend your answer.

Answer: The limit is 0. The "damping" of the oscillations squeezes the function values toward 0. (This example is a good one to illustrate the "squeeze theorem" or "sandwich theorem" for limits.)