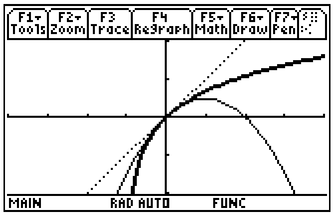
Linear and Quadratic Approximations

By Dave Slomer

What do a line, a parabola, and a log function have in common?

Fig. 1 

In figure 1, the dark curve is $y_1 = \ln(x+1)$. The dotted graph is the line $y_2 = x$. The other graph is the parabola $y_3 = x - \frac{x^2}{2}$. Enter those (linear and quadratic) functions into your TI-89 in the window [-4,4] by [-2,2] (as shown by the tick marks on the axes in figure 1).

Exercise 1: Find the point where the graphs of all 3 functions meet.

Exercise 2: Find the slope of each function at the point that they have in common.

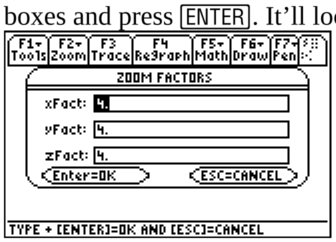
Of course, the second derivative for any line is constantly 0 (Why?), but not for parabolas and log functions.

Exercise 3: Compute the second derivative of the parabola and the log function at their point of intersection.

So, the 3 functions have a few properties in common.

So what?

So, set your "x- and y-zoom factors" to 4 and zoom in at (0,0), the point of intersection of all 3 graphs. [To do so, press ◆ F1 ([Y=]) or ◆ F2 ([GRAPH]) and then press F2 (**Zoom**) ⬆ (**C: Set Factors**) ENTER, which brings up the **ZOOM FACTORS** screen. Put 4 into both the **xFact** and **yFact** boxes and press ENTER. It'll look like what's in figure 2. Then press F2 2 (zoom in) ENTER.]



Exercise 4: What (if anything) do you see that seems significant about the relationships among the graphs?

Exercise 5: Zoom in again. *Now* what of significance do you see?

Exercise 6: Zoom in again. What's it look like now, after 3 zoom-ins?

Approximating is an important fact of life in the real world. Calculus gives insight into many ways of approximating things: secant lines approximate tangent lines; average rates of change approximate instantaneous rates of change; rectangle sums approximate areas of regions with curved boundaries and the total change in some changing quantity; and now this.

Set up your '89's table to show numerically what is in the last graph window (after the third zoom-in, the window should be [-.0625,.0625] by [-.03125,.3125]). It would be reasonable to make **tblStart** ≈ **xmin**, about -.06 and **Δtbl** ≈ **Δx**, about .001. You will see what's in figure 3. [To set up the table, press [◆] [F4] ([TblSet]). To see it, press [◆] [F5] ([TABLE]). To make it so you can see all 3 functions' values, while viewing the table press [◆][|] and adjust the **Cell Width** by pressing (▶) and moving the cursor to the number of digits you'd like to see. See figure 4.]

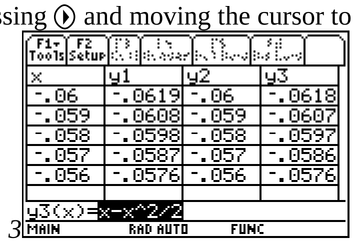


Fig. 3

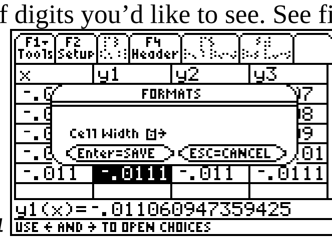


Fig. 4

[*Note: The* **x** *column of the table* ***is*** **y2**! *On the* [Y=] *screen, you might turn off* **y2** {by moving the cursor to it and pressing [F4]} *and read the* **y2** *values from the* **x** *column so more digits can be displayed in the table.*]

Exercise 7: In figure 3, you see that the line (**y2**) is not as good an approximation of the log (**y1**) as is the parabola. Noting the last line of the table in figure 3, move down through the table until the last **x** in the table (-.056) moves to the top line. Describe the relative accuracy of the linear and quadratic approximations that you now see—tenths? hundredths? thousandths? For each approximation, you might say something like, "Within ___ units of 0, the function *y* = ______ approximates *y* = log(*x*+1) with accuracy to the nearest ___." *(Hint: it's hard to mentally calculate the difference between* **y1** *and the approximating functions. You might define* **y4** *and* **y5** *to be the magnitude of the differences:* **y4=abs(y1(x)-y2(x))** *and* **y5=abs(y1(x)-y3(x))**, *turn off* **y1** ***through*** **y3**, *and look only at* **y4** *and* **y5** *instead. Using cell width of 8 will help. Refer to figs. 5 and 6.)*

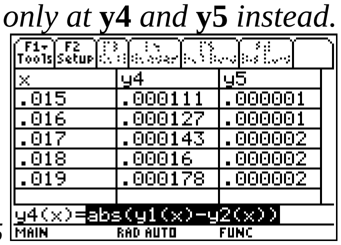


Fig. 5

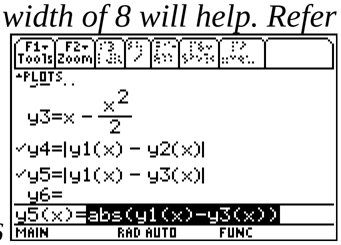


Fig. 6

Exercise 8: There are positive numbers involved in the graph window that you last zoomed to (in Exercise 6). Move to the other end of the "table window" by making **tblStart** = .06 (**xmax**, roughly) and **Δtbl** = –.001 (note: negative .001). Make similar observations as in Exercise 7.

Computations take time. Time is money. Fewer computations mean more money. (Graph that!) So sometimes (for other functions or some applications in the "real world") it might somehow be better to use the linear approximation, even though it's not as good as far away from the approximation point.

Exercise 9: Suppose you need accuracy to ten-thousandths. How far away from 0 (to the left and the right) would you be able to go so that the linear approximation would do?

Exercise 10: Same question as Exercise 9, but for the parabola (quadratic approximation).

Exercise 11: Slightly adjust your answers to Exercises 9 and 10 so that the "range of accuracy" can be stated as a slightly more restrictive absolute value inequality of the form $|x - 0| <$ _____.

Exercise 12: Find linear and quadratic approximations for the exponential function $y = e^x$ at (0,1). To do so, make sure that the first derivative of the line and the first and second derivatives of the parabola equal the first and second derivatives of the exponential function, **all at *x*= 0**. (This is how the 2 approximating functions in figure 1 were found.)

Store your linear approximation into **y2** and quadratic into **y3**.

Exercise 13: Give an absolute value inequality (as in Exercise 11) that will describe the values of x for which the linear approximation of the exponential function is accurate to thousandths.

Exercise 14: Same as Exercise 13, but for the quadratic approximation.

Exercise 15: Why is the **linear** approximation always going to be the **line** that is **tangent** to the function that it approximates at the point where the approximation is being done?

Exercise 16: Do you think "tangent parabola" is a good way to describe the relationship between the quadratic approximation and the function being approximated at the given point? Do you think you could find a 2nd parabola that is tangent to the exponential function at (0,1)? Share your thoughts.

Exercise 17: Try to find linear, quadratic, and *cubic* approximations to the sine function ($y = \sin(x)$) at the origin. What equations did you get for the 3 approximations? What do you think accounts for the fact that you couldn't find 3 different approximations? Graph what you did get in a window that best shows both the good and the bad of the approximations. This might help explain why two of the approximations were the same.

Exercise 18: The cosine function ($y = \cos(x)$) has an interesting property. Find its linear approximation at (0,1). Graph both in a window containing points not too far away from

(0,1). Are you surprised by the result? For which values of x is the linear approximation accurate to tenths?

What you have been doing borders on the somewhat advanced topic “Taylor Polynomials.” I hope you have time to study it sometime!

Calculus Generic Scope and Sequence Topics: Derivatives, Applications of Derivatives
NCTM Standards: Number and operations, Algebra, Geometry, Measurement, Problem solving, Connections, Communication, Representation