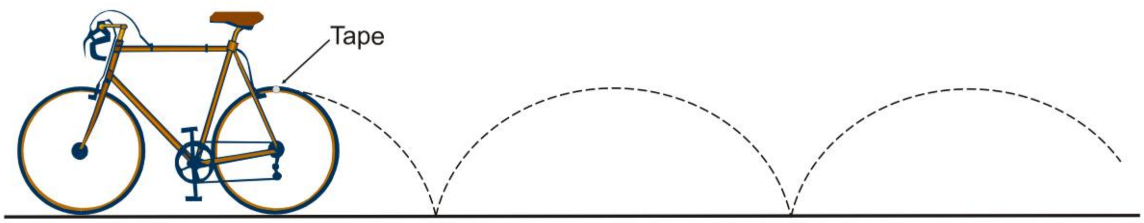
NUMB3RS Activity: Cycloid II
Episode: "The Mole"

Topic: Pre-Calculus and Calculus **Grade Level:** 10 - 12
Objective: Approximate the arc length of a cycloid
Time: 10 - 15 minutes

Introduction

In "The Mole," Charlie helps the FBI analyze a hit-and-run accident involving a woman and a car. To better understand the situation, Charlie examines the mechanics of walking. Charlie states that "when you walk, it's really a series of little circles rotating inside a larger circle. The heel orbiting backwards, then forward past the knee is a small circle within the larger circle of walking." The objective then becomes to determine the path of the heel not only as it moves in the circle, but as the person walks forward.

Imagine that Charlie attaches a piece of reflective tape to the tire of Larry's bicycle. As Larry rides his bicycle, Charlie plots the path of the tape. The path that it follows is called a cycloid, which is a combination of translation (the bike moving forward) and rotation (the wheel turning).



Discuss with Students

This activity introduces the concept of limits and infinite divisibility. Discuss with students the concept of finding a midpoint and that no matter how small the segment is, a midpoint always exists. It also may be helpful to discuss the paradox of Achilles and the Tortoise (see the Extensions section). Both of these examples will assist the students in making the transition to speculating what happens when there are infinite segments to approximate the arc length.

Student Page Answers:
1. *The lateral distance between a set of cusps*. **2.** *The third one.* **3.** *The more segments the better they approximate the arc.* **4**. *The difference between the approximation and the actual length approaches zero.*

Extensions Answers:

1. $= \int_0^{2\pi} \sqrt{1-2\cos t + \cos^2 t + \sin^2 t}\; dt = \int_0^{2\pi} \sqrt{2-2\cos t}\; dt = \int_0^{2\pi} 2\left|\sin 2t\right|\, dt = 8\int_0^{\pi/4} 2\sin 2t\; dt$
$= 8\left(\cos 2t\right)\Big|_0^{\pi/4} = 8$

2. $8r$

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Name: ________________________________ Date: _______________

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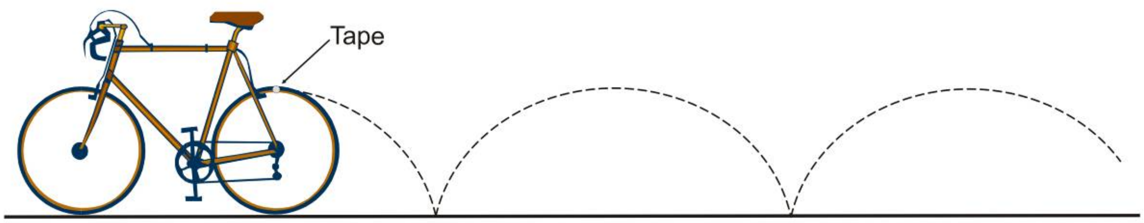
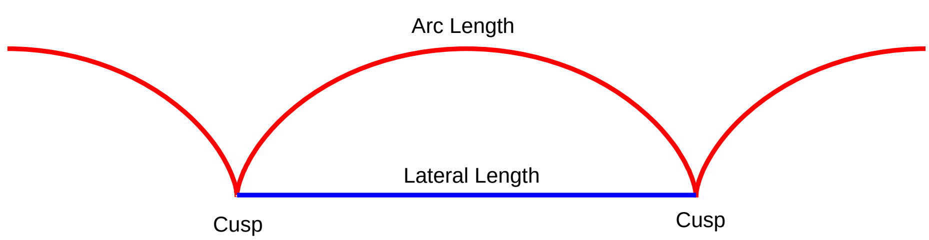


To complete his analysis and calculate the speed of the reflective tape on the wheel, Charlie must determine the length of the arc. Because this is a curved surface, we must approximate the arc length.

Each point where the cycloid touches the ground is called a cusp; the sections between the cusps are called humps. Charlie focuses on analyzing a single hump because the arc length of the entire cycloid is a multiple of the length of a single hump. The lateral length, arc length, and area under the curve can all be expressed in terms relating to the original circle and written in terms of the radius.



1. Is the circumference of Larry's wheel the arc length of a single hump or the lateral distance between a set of cusps?

You can approximate the arc length by deconstructing the curve as a series of straight lines, producing line segments that you can measure and find their sum. This process previews several concepts in calculus such as the limit, sum of a sequence, and the fundamental theorem of calculus.

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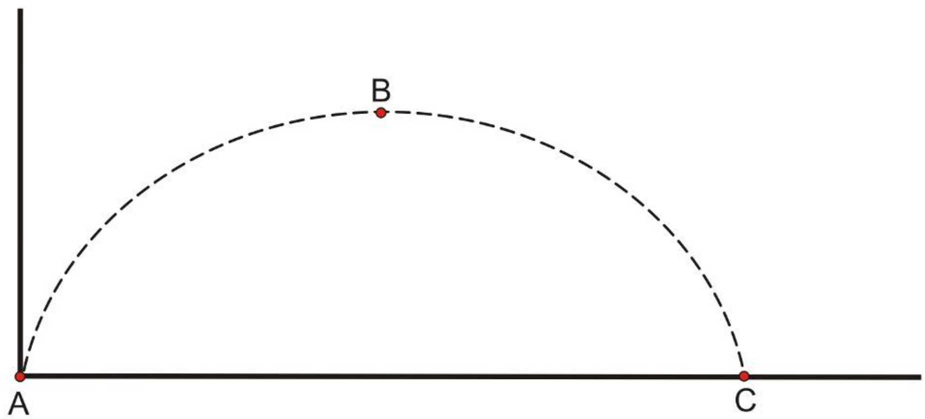
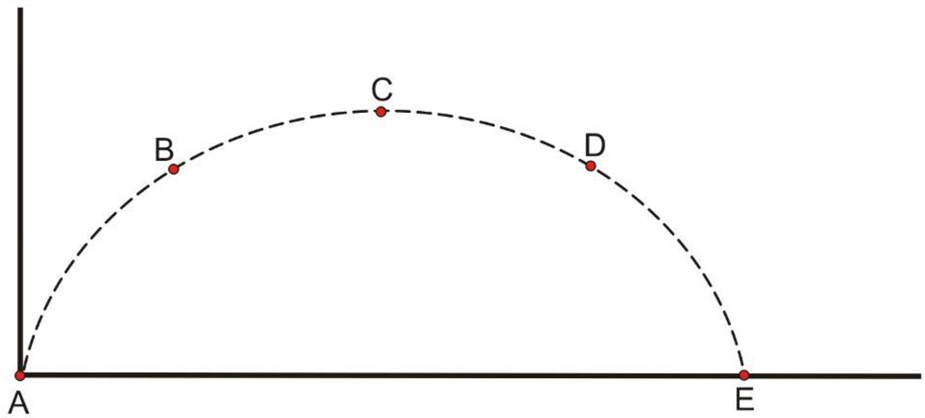
Draw the line segments for the arcs below, measure them, and add their lengths.



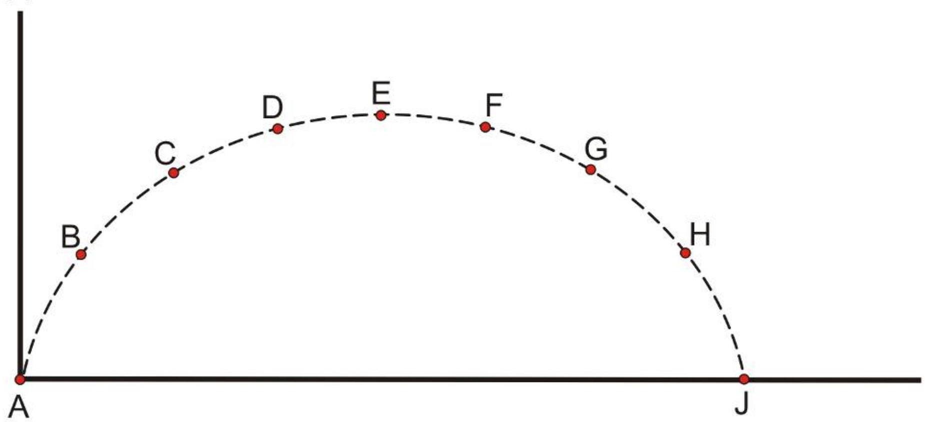
AB =
BC =

Sum =



AB =
BC =
CD =
DE =

Sum =



AB =
BC =
CD =
DE =
EF =
FG =
GH =
HJ=

Sum =

2. Which answer above is the best of the three approximations to the arc length?

3. Why would the number of segments make a difference in approximating the arc length?

4. As the number of segments approach infinity what can you say about the difference between sum of the segment lengths and the actual arc length?

The concept of using smaller and smaller intervals to approximate a curve's length is an important idea in calculus. This concept can be further explored in the Extensions section of this activity.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Introduction

Charlie tells us that cycloids are "rectifiable curves" because they have a finite length. Because of this observation together with the fact that the curves are differentiable (meaning that the curve is "smooth"), we can use the following formula to compute the length of the arc from point *a* to point *b*:

$$\text{ArcLength} = \int_a^b ds,$$

where *ds* represents the length of one of the small line segments used to approximate the length of the curve. Each of these segments is "infinitesimally" small, so that its length is very close to the section of the curve with the same endpoints. The symbol $\int$ represents summing up these small lengths. This sum is known as an integral, and is one of the key concepts in calculus.

In our example, the cycloid generated from a circle with a radius of 1 can be expressed with the parametric equations at the right, where (x, y) represents the location of the wheel at time t.

$$x = t - \sin t$$
$$y = 1 - \cos t$$

Now, set up the definite integral: $\text{Arc Length} = \int_a^b ds$.

From the distance formula we know that $ds = \sqrt{(dx)^2 + (dy)^2}$. Here, *dx* and *dy* denote the infinitesimal changes in x and y, respectively. The arc length can be expressed as shown at the right.

$$\text{Arc Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$
$$= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\ dt$$

1. Simplify the algebraic expression within the square root and if you know how to compute integrals, determine the exact arc length of a single hump of a cycloid.

2. With the arc length for a circle with radius 1 calculated; generalize the arc length of a single hump of a cycloid. Write your answers in terms of the radius of the circle.

Additional Resources

- The applet on this Web site computes the length of a broken-line approximation to a given curve: **http://xanadu.math.utah.edu/java/ApproxLength.html**
- To view calculus animations with created with Mathcad, go to **http://www.math.odu.edu/cbii/calcanim**.
- This Web site provides an explanation of Zeno's Paradox of the Tortoise and Achilles: **http://www.mathacademy.com/pr/prime/articles/zeno_tort**.
- In 1658, Sir Christopher Wren used a method of exhaustion to calculate the arc length of a cycloid. Read more about Sir Christopher Wren and the history of the cycloid at **http://www-groups.dcs.st-and.ac.uk/~history/Curves/Cycloid.html**.

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