EXPLORATIONS

Chapter 7

Nonlinear Pendulum Problems

In this chapter, you will explore the nonlinear model for a simple pendulum. The physical realization of a pendulum is a weight attached to one end of a rigid rod swinging from a fixed pivot at the other end of the rod.

Introduction

The standard model for a pendulum (assuming the mass of the rigid rod is negligible) is

$$\ddot{\theta} = -\frac{g}{l}\sin\theta, \quad \theta(t_0) = \theta_0, \quad \dot{\theta} = \alpha_0 \qquad \text{[Nonlinear Pendulum]}$$

where θ is the angle of deflection of the pendulum from vertical, g is the acceleration due to gravity, and l is the length of the rigid rod. The single dot above the function is Newton's notation for a derivative. In mathematical physics and engineering, you use this notation (and the double dot for a second derivative) exclusively for *time* derivatives. It is then common to say that for small angles, you can use the approximation $\sin\theta \approx \theta$ to work with the linear second-order equation that gives simple harmonic motion as a solution.

$$\ddot{\theta} = -\frac{g}{l}\theta, \quad \theta(t_0) = \theta_0, \quad \dot{\theta} = \alpha_0 \qquad \text{[Linear Pendulum]}$$

Here you will explore how the solutions to these two problems differ, both when the "small angle assumption" is valid and when it is not.

Example 1: The Small Angle Assumption

For simplicity, suppose that the pendulum is released in the position of a small angle from the vertical, for example, $0 < \theta_0 \le 15°$, $\alpha_0 = 0$.

Note: *It is common to give measurements in degrees because this means of measurement is easier to visualize. Radian measurement, however,* ***must*** *be used in the actual computations for the differential equation here.*

Assume that the length of the pendulum is such that $g/l = 2$. Then it is well known that the solution for the linear pendulum problem is

$$\theta(t) = \theta_0 \cos\left(\sqrt{2}\,t\right).$$

Look at the solution of the nonlinear problem for a short time span graphically and look at the solution for a long time span numerically and compare the results.

Solution

Look at θ_0 equal to 5°, 10°, and 15°.

1. Place the initial angles (in radians) that you wish to consider in a list and store to **T0** (T zero).

   $\{5*\pi/180,\ 10*\pi/180,\ 15*\pi/180\}$ [STO▸] T0

   Enter the second order nonlinear pendulum equation as a first-order system. (Figure 7.1)

Figure 7.1

2. Use the list you have just stored as the initial condition for **QI1** by typing the name **T0** there. The list will appear in the **QI1** field. Then enter a list of zeros for **QI2** of the same length. (Figure 7.2)

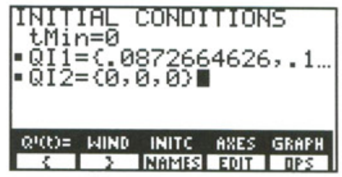

Figure 7.2

3. Adjust all of the other settings as shown in Figures 7.3 through 7.6.

   Each looks very much like the graph of a multiple of

   $$\cos\left(\sqrt{2}\,t\right)$$

   In particular, you expect the first zero at about

   $$t = \sqrt{2}\,\pi/4 \approx 1.111072073454$$

   which you store as *A*. Then you expect another zero if you add a multiple of

   $$\sqrt{2}\,\pi/2 \approx 2.22144146908$$

   which you store as *B*. The zeros in the plot should be approximately 1.11072, 3.33216, 5.55360, 7.77505, and 9.99649.

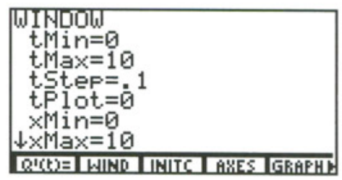

Figure 7.3

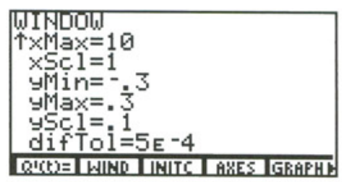

Figure 7.4

Figure 7.5

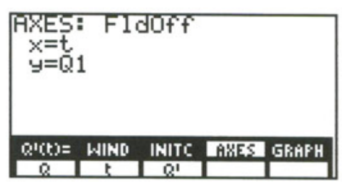

Figure 7.6

4. Display the plot of the solutions. (Figure 7.7)

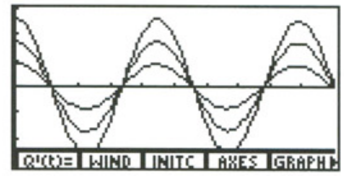

Figure 7.7

5. To speed up the computations, change the initial conditions to be **QI1** = 15* π /180, **QI2**=0. You can then evaluate the solution to the differential equation either in the home screen (getting the command from the CATLG menu) or in the graph screen (getting the menu command with the [MORE] key). (Figure 7.8)

Figure 7.8

The command **eval** only accepts an argument between **tMin** and **tMax**. Thus, to check for longer time, change **tMax** = 100 and compute

eval (A+20*B) {.044208678652 $^{-}$.314114369313}

eval (A+40*B) {.060256684912 $^{-}$.259491122303}

All of these evaluations give a first term in the list that is acceptably close to zero, given our tolerance for numerically solving the differential equation. The second term is the derivative *Q2* evaluated at the same point, and this always has absolute value near 0.37, which is close to $\theta_0\sqrt{2}$. (Figures 7.9 and 7.10)

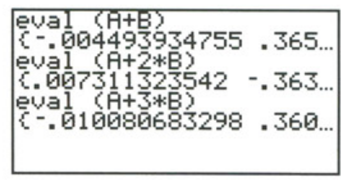

Figure 7.9

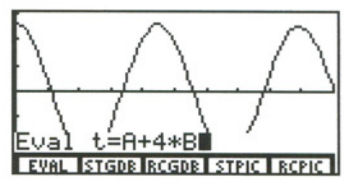

Figure 7.10

6. Another way to see the circular nature of the solution in this situation is to look at a phase diagram, changing the axes editor ([F4] **AXES**) to **x**=**Q1**, **y**=**Q2**, change the differential equation format screen ([F1] **FORMT**) to **DirFld**, and change the differential equation window editor ([F2] **WIND**) to **tMin**=0, **tMax**=10, **tStep**=.1, **tPlot**=0, **xMin**= $^{-}$.5, **xMax**=.5, **xScl**=.1, **yMin**= $^{-}$.4, **yMax**=.4, **yScl**=.1, **difTol**=5E $^{-}$4. Leave the initial condition as **QI1**=.26179938779915, **QI2**=0. (Figure 7.11)

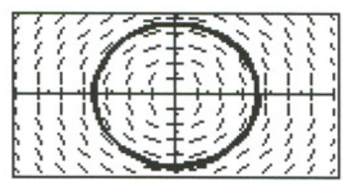

Figure 7.11

Example 2: Pendulum Problems with Angles of 30° to 60°

Compare the solutions to the nonlinear and linear pendulum problems for initial angles between 30° and 60°, or

$$0.524 \le \theta_0 \le 1.047 .$$

Continue to assume that the initial rate of change of the angle is zero (released) and that $g/l = 2$.

Solution

1. Keep the nonlinear differential equation from Example 1 in the variables **Q'1** and **Q'2**. To compare the graphs of the solutions, add the linear pendulum equations to the system. The added equations **Q'3** and **Q'4** are uncoupled from the first two equations (and could be solved independently). The TI-86 will numerically solve this as one big system.

   You will need to remember that even when only one part is plotted, there needs to be similar initial conditions for all four unknowns. You have changed the style for the last two equations to thin in Figure 7.12. See Figures 7.12 through 7.20 as you compare direction fields and (later) solutions. Note that the initial conditions fields remain blank. (Figure 7.16)

Figure 7.12

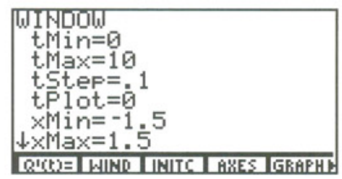

Figure 7.13

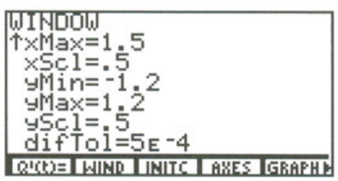

Figure 7.14

Figure 7.15

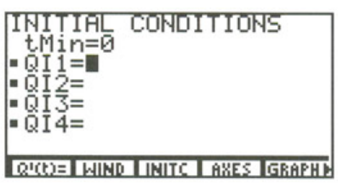

Figure 7.16

Figure 7.17

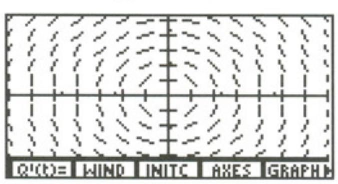

Figure 7.18 (Nonlinear)

Figure 7.19

2. Direction fields show us little difference yet, but the solutions will differ a little more if you change the differential equation format screen (**FORMT**) to **FldOff**, and set initial conditions to be **QI1**={0.6, 1}, **QI2**={0, 0}, **QI3**={0.6, 1}, and **QI4**={0, 0}. Change the x-range in the window editor to **xMin**=0, **xMax**=10, **xScl**=1.

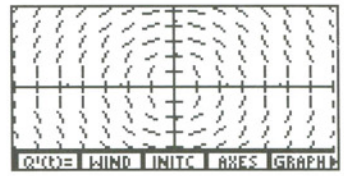

Figure 7.20
(Linear)

3. Plot with axes **x**=t and **y**=Q3. (Figure 7.21)

   Save this as a picture with a **STPIC** command.

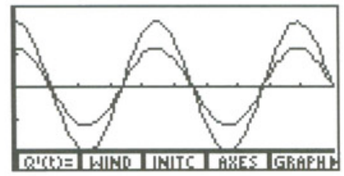

Figure 7.21

4. Plot with axes **x**=t and **y**=Q1. (Figure 7.22)

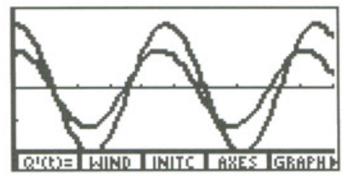

Figure 7.22

5. Press [MORE] [MORE] [F5] (**RCPIC**) to recall the picture from Step 3 (Figure 7.23) where you see that the linear and nonlinear models start to differ, even after only a few swings.

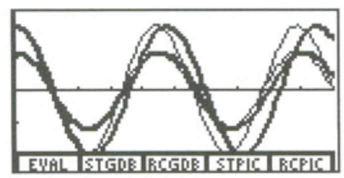

Figure 7.23

Note how the increasing initial amplitude is changing the period of the solutions in the nonlinear model, but not in the linear model. You can see this difference even more if you look numerically at longer times, as you did in Example 1. A and B have the same values as in Example 1 because the period of the linear model solution does not change with amplitude. First change to **tMax**=50, **QI1**=1, **QI2**=0, **QI3**=1, and **QI4**=0. Then you can compute the following (reporting only six decimal places).

eval A	{0.099530 ⁻1.347834 ⁻0.000169 ⁻1.412937}
eval (A+4*B)	{0.768173 ⁻0.833252 ⁻0.001615 ⁻1.402558}
eval (A+20*B)	{0.668089 ⁻0.940112 ⁻0.007239 ⁻1.361533}

Theoretically, the third item each time, Q3, should be zero. The fourth item each time should be

$$-\sqrt{2}\,.$$

These computed numbers exhibit normal roundoff errors for a numerical solution. The first and second items on each line above are just as accurately computed.

Example 3: Pendulums with a Push

Suppose that you give the pendulum a big push, releasing it just as it passes the vertical with $\theta_0 = 0$. Assume that $g/l = 2$ so that you can continue to use the same equations in the differential equation editor. Experiment with how the size of **QI2=QI4** affects the nonlinear and linear solutions.

Solution

1. Begin by choosing initial conditions θ_0'=**QI2=QI4**=1 and θ_0= **QI1=QI3**=0, with **yMin**= ⁻1 and **yMax**=1 as shown in Figure 7.24. Then change θ_0'=**QI2=QI4** to the values 1.5 (Figure 7.25, **yMin**= ⁻1.5, **yMax**=1.5), 2 (Figure 7.26, **yMin**= ⁻2, **yMax**=2), 2.8, 2.84, and 2.9 (Figures 7.27 through 7.29, **yMin**= ⁻3, **yMax**=10).

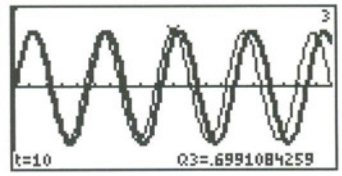

Figure 7.24

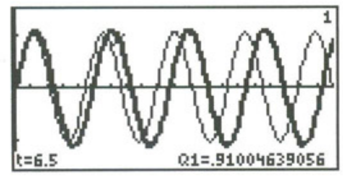

Figure 7.25

2. Figures 7.24 through 7.29 have axes **x**=t and **y**=Q with only equations **Q'1** (thick) and **Q'3** (thin) selected. In all of the plots, **tMin**=0, **tMax**=20, **xMin**=0, **xMax**=20, and **difTol**=5E⁻4.

   In Figures 7.28 and 7.29, the nonlinear model (thick) actually flips over the top. Since there is no friction or air resistance in the model, the pendulum will just keep flipping over the top forever.

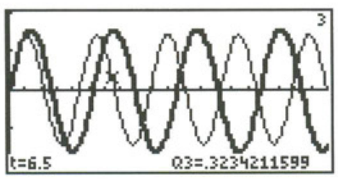

Figure 7.26

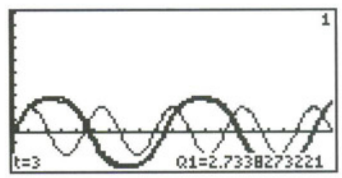

Figure 7.27

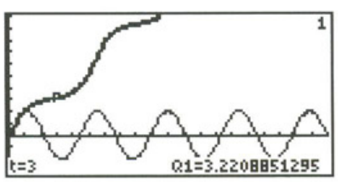

Figure 7.28

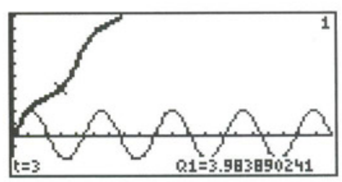

Figure 7.29

Exercises

1. An old-fashioned grandfather clock uses a pendulum to give it a regular periodic time interval with which to measure the passage of time. The weight of the pendulum can be adjusted up and down the rod to change the length of the pendulum *l*. Compute both the linear and nonlinear pendulum model in Example 1, keeping the same small initial angle θ_0 and $\alpha_0 = 0$ but using different vales for *g/l* from 1.5 to 2.5 (which would correspond to lengthening or shortening *l*).

2. In Example 2, expand the window in Figure 7.18 and use the **EXPLR** feature to investigate various solutions for initial conditions of the magnitude considered in that example.

3. There is a critical value for θ_0 between 2.8 and 2.84 when the pendulum changes from returning to the starting position in Example 3 or "flipping over the top." Determine more accurately where this happens (at least one more digit) and explore what the solution looks like very near this value.

4. Consider adding a term to the nonlinear pendulum equation to represent air resistance to the motion of the pendulum through the air or friction on the pivot (linear damping).

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - k\dot{\theta}, \quad \theta(t_0) = \theta_0, \quad \dot{\theta} = \alpha_0 \quad \text{[Nonlinear Pendulum with Linear Damping]}$$

Explore the effect of a friction constant $k = 0.03$ or 0.5 on the initial conditions from Example 1.