Pythagoras is Not Real

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Teacher Answers

7 8 9 10 **11** 12

TI-Nspire™ | Investigation | Student | 25 min

Introduction

Don't be alarmed, this is not the same situation as one of the fictional entities that visit your house annually to deposit chocolate eggs or place gift wrapped items under a tree. This story is about the retrospective beauty of mathematics. Pythagoras definitely existed; planes and imaginary numbers[1] did not. Pythagoras was born around 569 BC, more than 2000 years later, complex numbers evolved. The retrospective beauty involves a lovely relationship that exists between complex numbers and our endless quest for the delightful trios that continue to contribute to the fame of Pythagoras.

Warm Up Questions

Question: 1.

Determine each of the following (by hand). Use the calculator to check your answers.

a. $z = 2 + i$ calculate: z^2

$$(2+i)(2+i) = 4 + 4i + i^2$$
$$= 3 + 4i$$

b. $z = 3 + 2i$ calculate: z^2

$$(3+2i)(3+2i) = 9 + 12i + 4i^2$$
$$= 5 + 12i$$

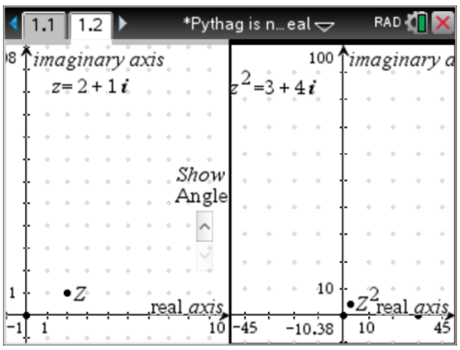
c. $z = 5 + 2i$ calculate: z^2

$$(5+2i)(5+2i) = 25 + 20i + 4i^2$$
$$= 21 + 20i$$

d. $z = 4 + 3i$ calculate: z^2

$$(4+3i)(4+3i) = 16 + 24i + 9i^2$$
$$= 7 + 24i$$

TI-nspire Investigation



Open the TI-nspire file: "Pythag is not Real"

Navigate to page 1.2.

The calculator screen has been split into two Graph applications. Each graph application contains an Argand plane, one for z (left) and the other for z^2.

The number can be moved around the plane. For the purposes of this exercise both the real and imaginary components are positive integers.

The value for z^2 is displayed on the Argand plane on the right hand side of the screen. **The scale is different for the right-hand plane as the z^2 values have a much larger magnitude.**

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[1] Imaginary Numbers – Heron of Alexandria, born more than 500 years after Pythagoras may have conceived the idea of an imaginary number, however it was not until 1572 that Rafael Bombelli authored: l'Algebra, a set of three books that the first represented an imaginary number in the form: $\sqrt{-1}$

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Question: 2.

The calculator screen displayed (above) shows: $z = 2 + i$ and the corresponding value for $z^2 = 3 + 4i$.

a. Determine the value of $\left|z^2\right|$ (Magnitude of z^2). $\left|z^2\right| = \sqrt{9+16} = 5$

b. State the Pythagorean triple that z^2 relates to where $z = 2 + i$ 3, 4, 5

Question: 3.

Drag z to the point: $z = 3 + 2i$ and determine the corresponding value for z^2.

a. Determine the value of $\left|z^2\right|$ (Magnitude of z^2). $\left|z^2\right| = \sqrt{25+144} = 13$

b. State the Pythagorean triple that z^2 relates to where $z = 3 + 2i$ 5, 12, 13

Question: 4.

Drag z to the point: $z = 5 + 2i$ and determine the corresponding value for z^2.

a. Determine the value of $\left|z^2\right|$ (Magnitude of z^2). $\left|z^2\right| = \sqrt{441+400} = 29$

b. State the Pythagorean triple that z^2 relates to where $z = 5 + 2i$ 20, 21, 29

Question: 5.

Let $z = a + bi$ such that: $b > a > 0$. Create three more imaginary numbers and check if Pythagorean triples are produced through: z^2.

Answers will vary; some will be multiples of other Pythagorean triples previously generated. For Pythagorean triples having the real component greater than the imaginary component means that arg(z) < $\pi/4$, which means that arg(z^2) < π resulting in positive real and imaginary components.

Question: 6.

Let $z = a + bi$ show that real and imaginary components of z^2 will always produce the shorter two sides required for a Pythagorean triple.

$$z^2 = a^2 - b^2 + 2abi$$
$$\therefore \quad \mathrm{Re}(z^2) = a^2 - b^2$$
$$\therefore \quad \mathrm{Im}(z^2) = 2ab$$
$$\begin{aligned}\mathrm{Re}(z^2)^2 + \mathrm{Im}(z^2)^2 &= (a^2 - b^2)^2 + (2ab)^2 \\ &= a^4 + b^4 - 2a^2b^2 + 4a^2b^2 \\ &= a^4 + 2a^2b^2 + b^4 \\ &= (a^2 + b^2)^2\end{aligned}$$

The final term is a perfect square which shows that $\mathrm{Re}(z^2)^2 + \mathrm{Im}(z^2)^2 = (a^2 + b^2)^2$

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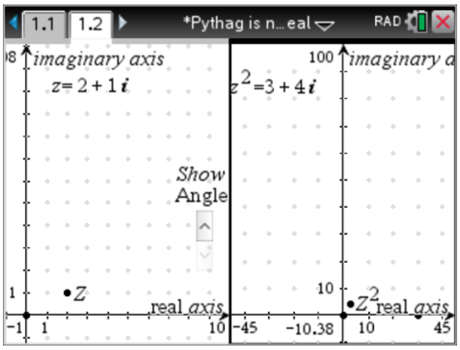
Extension Questions

The slider on page 1.2 can reveal the angle z makes with the real axis and also the magnitude of z.

- Click once on the slider to reveal the angle.
- Click on the slider a second time to reveal the magnitude
- Click on the slider a third time to reveal both the angle and the magnitude.



These measurements are dynamic. Move z around and look for patterns relating:

- The angle z makes with the real axis and the angle z^2 makes with the real axis.
- The magnitude for z and the corresponding magnitude of z^2.
  Hint: Try values such as; $z=4+3i$ and $z=3+4i$.

Question: 7.

What 'pattern' exists between the angle z makes with the real axis and z^2 makes with the real axis?

The angle that z^2 makes with the real axis is double the angle that z makes.

Question: 8.

What 'pattern' exists between the magnitude of z and z^2.

The magnitude of z^2 is the square of the magnitude of z.