Constructing an Ellipse

ID: 9979

Time required
60 minutes

Activity Overview

In this activity, students will explore two different methods for constructing an ellipse. Students discover that the sum of the distances from a point on an ellipse to its foci is always constant. This fact is then used as the basis for an algebraic derivation of the general equation for an ellipse centered at the origin.

Topic: Analytic Geometry — Conics & Polar Coordinates

- *Derive the equation (in rectangular form) of an ellipse as the locus of a point that moves so that its total distance from two fixed points (–f, 0) and (f, 0) is a constant.*
- *Write the equation of an ellipse with center at (0, 0) given its vertices and co-vertices and graph it.*

Teacher Preparation and Notes

- *This activity is appropriate for an Algebra 2 or Precalculus classroom.*
- *Students should have experience using the distance formula and solving radical equations.*
- *This activity is intended to be* ***teacher-led*** *with students in* ***small groups****.*
- ***To download the TI-84 files (.8xv and .8xi files) and student worksheet, go to education.ti.com/exchange and enter "9979" in the keyword search box.***

Associated Materials

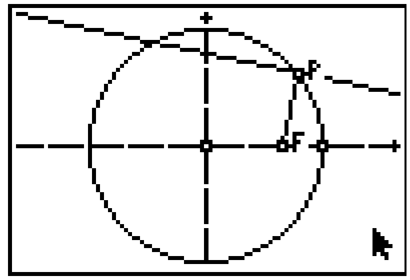
- *ConstructingAnEllipse_Student.doc*
- *ELLIPSE1.8xv*
- *ELLIPSE2.8xv*
- *Pic1.8xi*
- *Pic2.8xi*
- *Pic3.8xi*

Suggested Related Activities

To download any activity listed, go to education.ti.com/exchange and enter the number in the keyword search box.

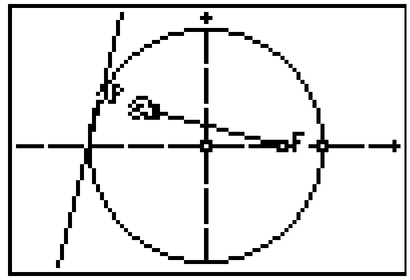
- *Orbit Of Jupiter (TI-84 Plus family) — 10034*
- *Ellipse: Envelope of Lines (Cabri Jr.) (TI-84 Plus family) — 7291*
- *Ellipse: Locus of Points (Cabri Jr.) (TI-84 Plus family) — 7290*
- *NUMB3RS – Season 2 – "Harvest" – Waxing Elliptical (TI-84 Plus family) — 6522*

Problem 1 – Envelope Construction

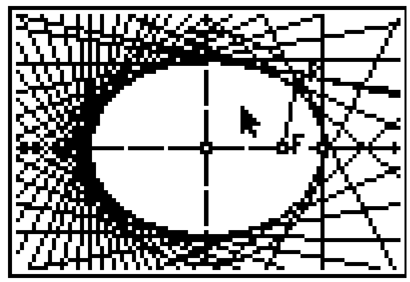


Studens should begin by opening *Cabri Jr.* from the application list. To open the first file, they should press **F1**, select **Open…,** and select *ELLIPSE1* from the list.

In the diagram, point *F* lies on a diameter of the circle, segment *FP* connects *F* to the point *P* (which lies on the circle), and a perpendicular line to segment *FP* is drawn.

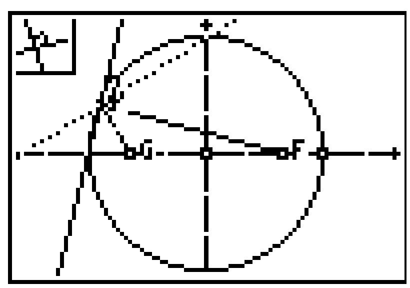


Students should drag point *P* around the circle. To drag the points, move the cursor over point *P* (It will turn white.), press ALPHA to "grab" the point, use the arrows keys to move it, and then press ALPHA again to let it go.

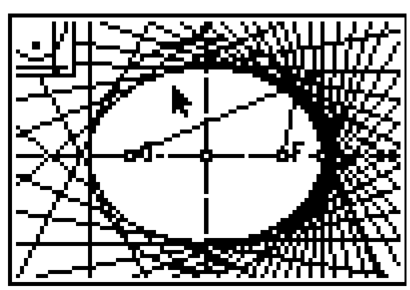


Students will next create the locus of perpendicular line as *P* moves along the circle. To create the locus, go to **F3 > Locus**. Click on the perpendicular line (*not* on point *P*) and then on point *P*.

This is one way to construct an ellipse called the "envelope method." Note that the diameter of the circle is equal to the width of the ellipse along its *major* axis (the longer of its two axes). The point *F* is a special type of fixed point that can be used to generate the ellipse. Ellipses have two such fixed points, called *foci* (singular: *focus*).



To explore the other focus, students should either delete or hide the existing locus using the **Hide/Show** or **Clear** tool. Then, they should reflect *F* over the *y*-axis, select **F4 > Reflection**, click on *F*, and then on the *y*-axis. Students will label this point *G* using **F5 > Alph-Num**. The image point should be labeled *G* using **F5 > Alph-Num**. Move the cursor over the point. (it will become larger). Press ENTER, then type *G*.



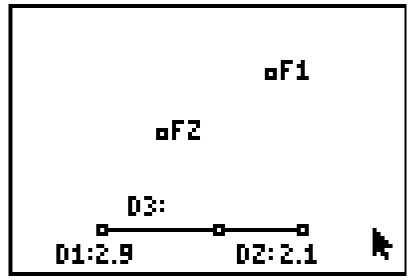
Segment *GP* should be constructed. Then, students will construct a similar setup to what was used for the other focus.

By dragging the focus *F* (which in turn moves *F'*), students should find that the location of the foci affects the shape of the ellipse.

Problem 2 – String and Pins Construction

An **ellipse** is defined as the set of points in a plane such that the sum of the distances from two fixed points (foci) in that plane is constant. Students will now use this definition to construct an ellipse.

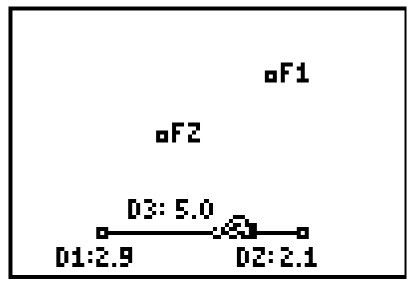
Students will need to use the *Cabri Jr.* file *ELLIPSE2*.



The diagram contains a segment with a slider and two additional points, *F1* and *F2,* which will become the foci of the ellipse.

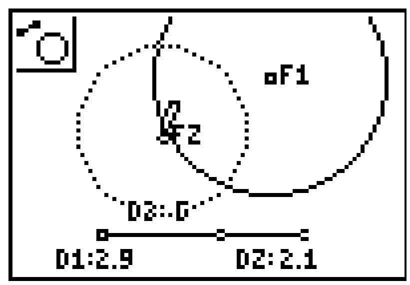
The values of *D1* and *D2*, determined by the slider, will be the distances from *F1* and *F2* (respectively) to the point on the ellipse.

Students should calculate *D3 = D1 + D2*. To do this, select **F5 > Calculate**, move the cursor over the first value, press ENTER, press +, move the cursor over the second value, press ENTER, move the cursor next to *D3*, and press ENTER to drop the value.

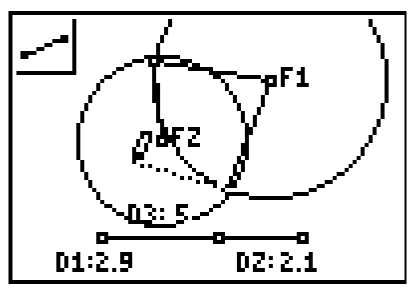


Next, students should drag the slider, changing *D1* and *D2* to see that the sum of *D1* and *D2* is always equal.

Students will draw a circle with center *F1* and radius *D1*. They should use **F3 > Compass**. To set the radius of the circle equal to *D1*, choose the left end of the segment, then to the slider. A dotted circle appears. To center the circle on *F1*, move the cursor to the point and press ENTER. They should repeat the process to draw a circle with center *F2* and radius *D2*.

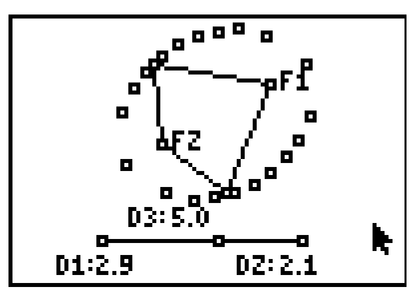


Next, students should mark the intersections of the circles (Go to **F2: Point > Intersection** and select the two circles.) and then construct four segments (the radii of the circles).



Mark the two intersection points of the circles. Go to **F2: Point > Intersection** and select the two circles. Then, hide the circles.

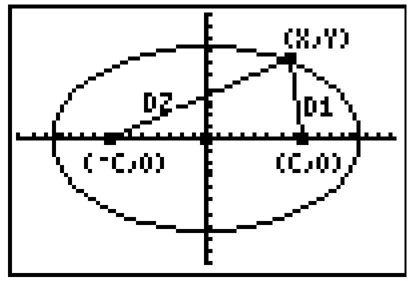
Create the loci of the two intersection points as the slider travels along the segment.



This construction is called the "string and pins" construction because it is traditionally performed by wrapping a piece of string (represented here by the segment and slider) around two pins driven into a flat surface at the foci.

Problem 3 – Deriving the Equation of an Ellipse

Students begin this problem by opening the picture file *PIC1*. On the Home screen, press [2nd] [DRAW], arrow over to the STO menu, select **2:RecallPic**. To enter the name of the picture, they can press [VARS], select **4:Picture…**, and select the variable name from the list.



The diagram shows an ellipse with a center at the origin and the beginnings of the derivation. Students will follow the steps to derive a general equation for such an ellipse.

The lengths of the segments connecting (X, Y) to the foci are *D1* and *D2*. Let C be the distance from the center to the foci.

To open another picture, students should first use the **ClrDraw** command (found in the Draw menu) to clear the diagram.

1. $2A = D1 + D2$

2. $D1 = \sqrt{(X-C)^2 + (Y-0)^2} = \sqrt{(X-C)^2 + Y^2}$

   $D2 = \sqrt{(X-(-C))^2 + (Y-0)^2} = \sqrt{(X+C)^2 + Y^2}$

3. $2A = \sqrt{(X-C)^2 + Y^2} + \sqrt{(X+C)^2 + Y^2}$

4. $\sqrt{(X+C)^2 + Y^2} = 2A - \sqrt{(X-C)^2 + Y^2}$

   $\left(\sqrt{(X+C)^2 + Y^2}\right)^2 = \left(2A - \sqrt{(X-C)^2 + Y^2}\right)^2$

   $(X+C)^2 + Y^2 = 4A^2 - 4A\sqrt{(X-C)^2 + Y^2} + (X-C)^2 + Y^2$

   $-4A\sqrt{(X-C)^2 + Y^2} = -4A^2 + (X+C)^2 + Y^2 - (X-C)^2 - Y^2$

   $\sqrt{(X-C)^2 + Y^2} = -\dfrac{1}{4A}(-4A^2 + (X^2 + 2XC + C^2) + Y^2 - (X^2 - 2XC + C^2) - Y^2)$

   $\sqrt{(X-C)^2 + Y^2} = -\dfrac{1}{4A}(-4A^2 + 4XC)$

   $\sqrt{(X-C)^2 + Y^2} = A - \dfrac{XC}{A}$

$$\left(\sqrt{(X-C)^2+Y^2}\right)^2=\left(A-\frac{XC}{A}\right)^2$$

$$(X-C)^2+Y^2=A^2-2XC+\frac{C^2}{A^2}X^2$$

$$X^2-2XC+C^2+Y^2=A^2-2XC+\frac{C^2}{A^2}X^2$$

$$X^2+C^2+Y^2=A^2+\frac{C^2}{A^2}X^2$$

5. $$X^2-\frac{C^2}{A^2}X^2+Y^2=A^2-C^2$$

$$X^2\left(1-\frac{C^2}{A^2}\right)+Y^2=A^2-C^2$$

$$X^2\left(\frac{A^2}{A^2}-\frac{C^2}{A^2}\right)+Y^2=A^2-C^2$$

$$X^2\left(\frac{A^2-C^2}{A^2}\right)+Y^2=A^2-C^2$$

6. $$X^2\left(\frac{A^2-C^2}{A^2}\right)+Y^2=A^2-C^2$$

$$\frac{X^2}{A^2}+\frac{Y^2}{A^2-C^2}=1$$

7. The segments connecting (*X*, *Y*) to the foci are the hypotenuses of two right triangles with two congruent sides, i.e., two congruent triangles. Since the triangles are congruent, we can conclude that *D1* = *D2*.

8. Recall that 2*A* = *D1* + *D2*. Therefore 2*A* = *D1* + *D1*, and *A* = *D1*.

9. Since *A* equals *D1*, $B^2=A^2-C^2$.

10. The expression in Step 6 can be written as $\frac{X^2}{A^2}+\frac{Y^2}{B^2}=1$.