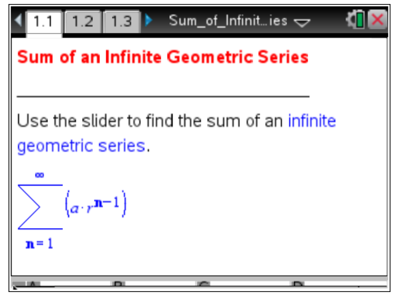
Sum of an Infinite Geometric Series



Math Objectives

- Students will understand how a unit square can be divided into an infinite number of pieces.
- Students will understand a justification for the following theorem: The sum of the infinite geometric series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=1$.
- Students will be able to explain why the sum of an infinite geometric series is a finite number if and only if $|r| < 1$.
- Students will use appropriate tools strategically (CCSS Mathematical Practice).
- Students will look for and make use of structure (CCSS Mathematical Practice).

Vocabulary

- geometric series
- infinite series
- ratio of a geometric series
- sigma notation

About the Lesson

- This lesson involves clicking on a slider to see that the area of a square that has been systematically divided into an infinite number of pieces approaches 1.
- As a result, students will:
  - Connect the area of a square with the sum of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$, and realize that the sum is 1.
  - Examine several infinite geometric series with various ratios to determine that the sum of an infinite geometric series is a finite number if and only if $|r|<1$.

TI-Nspire™ Navigator™

- Use Quick Poll to assess students' understanding.
- Use Class Capture to share students' formulas.
- Collect student documents and analyze the results.
- Utilize Class Analysis to display students' answers.

Activity Materials

- Compatible TI Technologies: TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software

Tech Tips:

- This activity includes screen captures taken from the TI-Nspire CX handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire App. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

Student Activity

- Sum_of_Infinite_Geo_Series_Student.pdf
- Sum_of_Infinite_Geo_Series_Student.doc

TI-Nspire document

- Sum_of_Infinite_Geo_Series.tns

Discussion Points and Possible Answers

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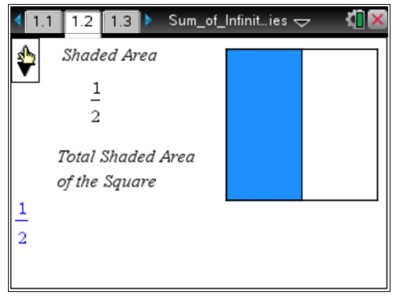


1. Select ▲ to see a rectangle with a shaded area of $\frac{1}{2}$ unit^2. The length of a side of the original square is 1 unit. What are the dimensions of the rectangle?

   <u>Answer:</u> The square measures 1 unit on each side. Since the rectangle contains a side of the square, that side must also measure 1 unit. The area of the rectangle is $\frac{1}{2}$ unit^2. Thus, $lw = 1w = \frac{1}{2}$, so we know that w must equal $\frac{1}{2}$. The dimensions of the rectangle are $1 \times \frac{1}{2}$.

2. Select ▲ until the shaded area increases to $\frac{7}{8}$. What are the dimensions of the three rectangles whose sum is $\frac{7}{8}$?

   <u>Answer:</u> The sum of the areas of the three rectangles is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$.

   We know from Question 1 that the first rectangle has dimensions $1 \times \frac{1}{2}$.

   The second rectangle is formed by halving the side of the first rectangle whose length is 1. Thus, the second rectangle has dimensions $\frac{1}{2} \times \frac{1}{2}$.

   The third rectangle is formed by halving the side of the second rectangle whose length is $\frac{1}{2}$. Thus, the second rectangle has dimensions $\frac{1}{4} \times \frac{1}{2}$.

3. What do you expect to be the area of the next rectangle to be added to the sum? (Express your area in both fractional and decimal forms.) Explain how you arrived at your conclusion.

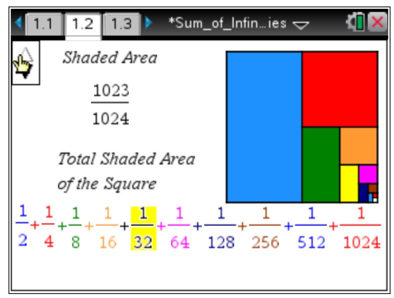
Answer: The next rectangle is formed by halving the side of the third rectangle whose length is $\frac{1}{2}$.

Thus, the fourth rectangle has dimensions $\frac{1}{4} \times \frac{1}{4}$. Its area is $\frac{1}{16}$.

The sum of the areas of the four rectangles is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}=0.9375$.

> **Teacher Tip:** By expressing their answers in decimal form, students can see how close the sum is getting to 1, even after only a few terms are used.

4. Continue selecting ▲ until you can't press it any more. If you could select ▲ again, what would the area of the next rectangle be? What would the total sum of the areas be? (Express your answers in both fractional and decimal forms.)



Answer: The next rectangle is formed by halving a side of the previous rectangle. This would make the area equal to half the area of the preceding rectangle.

Thus, the area of the new rectangle would be $\frac{1}{2}\cdot\frac{1}{1024}=\frac{1}{2048}$.

The area of the shaded region would be $\frac{1023}{1024}+\frac{1}{2048}=\frac{2047}{2048}\approx 0.9995$.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 1 at the end of this lesson.

5. If you could continue pressing ▲ an infinite number of times, and the whole region were shaded, what would the total shaded area be? (Express your answers in both fractional and decimal forms.)

Answer: Since the square has sides of length 1 unit, we know the area of the square is 1 unit^2. Thus, the sum of all of the rectangles that would completely shade the region must be 1 unit^2.

6. Write an expression for the sum of the areas of the infinite number of rectangles formed. What is the value of this sum? Why?

Answer: The sum of the areas of the infinite number of rectangles formed would be:
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$
Thus, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots = 1$.

7. Express your answer from Question 6 in sigma notation.

Answer: Expressed in sigma notation, we have $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 1$.

8. Instead of halving the side of the square, suppose that we doubled its size and continued to double a side of each subsequent square formed.
   a. Express the sum of the areas of these squares as an infinite sum.

   Answer: If we doubled the side of the square, we would have 2 + 4 + 8 + 16 +…

   b. What happens to this sum as the number of squares increases? Explain your answer.

   Answer: If we continued to increase a side of the square, the sum would get infinitely large.

9. Instead of halving the length or width of each of the rectangles, suppose that we multiplied the rectangle's length or width by $\frac{1}{3}$, giving us the series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots$
Do you think that the sum of the series would be finite or infinite? Explain.

Answer: If we continue to add $\frac{1}{3}$ of the previous term to the sum, the successive terms will continue to get smaller and the sum would approach a finite number. In this case, the sum is $\frac{1}{2}$.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 2 at the end of this lesson.

10. Give an example of an infinite geometric series that you think would have a finite sum and an example of one that you think would not have a finite sum. Explain your reasoning.

<u>Sample Answer:</u> For an infinite geometric series to have a finite sum, the common ratio (r) must be a proper fraction, i.e., $|r| < 1$.
Possible examples include 25 + 5 + 1 + 0.2 + … or 0.1 + 0.01 + 0.001 + 0.0001 + …

If $|r| \geq 1$, there would not be a finite sum. Possible examples include 1 + 4 + 16 + 64 + …

TI-Nspire Navigator Opportunity: *Class Capture*
See Note 3 at the end of this lesson.

11. Based upon the information above, what do you conjecture must be true about the ratio of the consecutive terms of an infinite geometric series for the series to have a finite sum?

<u>Answer:</u> The ratio, r, is conjectured to be a proper fraction, i.e., $|r| < 1$.

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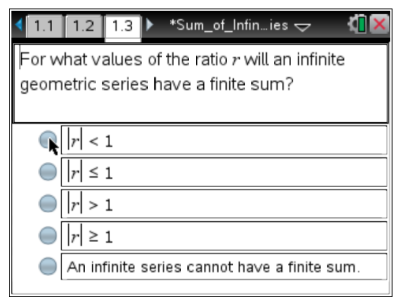

12. For what values of the ratio, r, does an infinite geometric series appear to have a finite sum?

<u>Answer:</u> The ratio r would appear to be a proper fraction, i.e., $|r| < 1$.

Tech Tip: This question can be answered on the worksheet or on Page 1.3 of the TI-Nspire document. It is a Self-Check question, enabling students to check if their answers are correct. With the TI-Nspire Navigator system, you can collect students' responses to analyze them.

Tech Tip: Students can self-check their answers to the included assessment questions by selecting 🔧 **> Check Answer**.

TI-Nspire Navigator Opportunity: *Class Analysis*
See Note 4 at the end of this lesson.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students are able to understand:

- An infinite geometric series can have a finite sum.
- The only values of the ratio, r, for which an infinite geometric series can have a finite sum are values of r such that $|r| < 1$.
- The area model for these infinite sums is not a proof. Examples do not constitute a proof. However, examples enable you to make conjectures that can then be proved.
- The activity in this lesson only addresses the sum of an infinite *geometric* series. You must be careful that students do not form the misconception that if the terms of *any* series approach 0, then the series converges. For example, the sum of a harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, diverges.

Extension

- You might want to utilize the principle of mathematics induction to prove that $1 - r^n = (1 - r)(1 + r + r^2 + r^3 + \ldots + r^{n-1})$.
- You can then have students determine that, for $|r| < 1$ as n approaches infinity, $1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}$.
- Students can generalize that for $|r| < 1$, $a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}$.

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Note 1

Questions 2, 3, 4, *Quick Poll*

You can send a Quick Poll to assess students' understanding of the process utilized to form each of the new rectangles.

Note 2

Question 9, *Quick Poll*

You can send a Quick Poll to determine whether or not students have reached the proper conclusion about the conditions under which an infinite geometric series has a sum. You can rephrase Question 9 as a true-false question, asking students if it's true that the sum would be finite. Be sure to ask students to explain their reasoning.

Note 3

Question 10, Class Capture

Ask students to insert a Calculator page into their document. They can type the first few terms of a geometric series with a finite sum.

Scroll through the various class captures, asking students what is common about the series. This will reinforce that the ratio must be a proper fraction.

Note 4

Question 12, *Learning Check and Class Analysis*

This question was created as a Self-Check question, enabling it to be utilized by students whether or not they have access to TI-Nspire Navigator. You might want to let students know that you will be collecting their answers at some point. If you choose to use students' scores for assessment, you can to change the question from a Self-Check to an Exam question.