NUMB3RS Activity: Chains and Pyramids
Episode: "Backscatter"

Topic: Geometric Progressions and Exponential Growth **Grade Level:** 8 - 12
Objective: To learn the mathematics behind chain letters and pyramid schemes and to understand why they are both dangerous and illegal.
Time: About 30 minutes
Materials: A calculator, a newspaper, a large calendar, and at least 15 pennies

Introduction
In "Backscatter," Charlie uses the mathematics of "Backscatter Analysis" to trace an Internet attack back to its source. While few home computer users are likely to become victims of an outright attack by Internet hackers, nearly everyone with an e-mail address is familiar with the problems that can be caused by "message flooding," the mathematically inevitable consequence of designing messages so that they will replicate themselves exponentially. Similar messages, called **chain letters**, have been clogging mail systems for nearly a century, often with the help of innocent victims who do not understand what they are doing. The Internet has only aggravated the problem by making the transmission of such letters faster and easier.

This activity will look at the mathematics of chain letters and pyramid schemes, both of which involve **geometric progressions** and **exponential growth**. Teachers can use the context of the activity to explain to their students that initiating chain letters or pyramid schemes is both *dangerous* and *illegal*, not only because they cause problems for the transmission system, but also because they usually involve fraud, potentially on a very large scale.

Discuss with Students
Ask students if they have ever received letters or e-mail messages that have asked them to "send this message to five of your friends" or "send this e-mail to everyone you know." Have any of them asked for money? Have any of them contained health warnings or computer virus alerts? Have any of them made threats if the "chain" is broken? A healthy discussion of the ethics of chain letters can debunk the idea that some chain letters are "good" and can set the scene for the mathematics in this activity.

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Student Page Answers:
1. *6* **2.** *36* **3.** *216* **4.** *Because the 6 people in the second generation are all friends of Notzo, they probably have many other friends in common. When each of them independently chooses 6 friends, they are very unlikely to be 36 different people.* **5.** *6^7 = 279,936 dollars* **6.** *6^{11} = 362,797,056* **7.** *Since the number of letters would have to be greater than the U.S. population, the chain will explode well before Notzo sees his first dollar.* **8.** *The most likely to profit is the first person in the chain, who will profit at the expense of the many people who come later, the victims of the fraud. The person who starts a chain letter can therefore be prosecuted for mail fraud.* **9.** *5, 25, 125, 625, 3,125, 15,625, 78,125, 390,625, 1,953,125, 9,765,625.* **10.** *Multiply by 5.* **11.** *$y = 5^x$* **12.** *An exponential function f on the domain of natural numbers defines a geometric progression. The nth term of the progression is f(n).* **13.** *Pyramid schemes, like chain letters, depend on the existence of a geometric progression of gullible participants. Inevitably, the progression becomes too big for the population to accommodate.*

More Fun Answers: *The tenth fold of newsprint involves bending 2^9 = 512 thicknesses of paper. On January 28 there will be a stack of 2^{27} pennies worth \$1,342,177.28. The stack on January 31 will be 2^{30} pennies tall. At about 16 pennies per inch, that is a stack 1,059 miles high.*

Challenge Answer: *In all, there will be $2^{31} - 1$ pennies on the calendar. That is more than 21 million dollars!*

Name: _________________________________ Date: ________________

NUMB3RS Activity: Chains and Pyramids

In "Backscatter," Charlie uses the mathematics of "Backscatter Analysis" to trace an Internet attack back to its source. One of the techniques used in such attacks is called "message flooding," which clogs the system along targeted paths, slowing down all transmissions in order to make the bad ones easier to trace. Ironically, while few home computer users complain of being attacked by hackers, nearly everyone with an e-mail address is familiar with message flooding – the mathematically inevitable consequence of e-mail messages that are designed to keep replicating themselves. Similar messages, called **chain letters**, have been clogging mail systems for nearly a century, often with the help of innocent victims who do not understand what they are doing. The Internet has only aggravated the problem by making the transmission of such letters faster and easier.

This activity will look at the mathematics of chain letters and pyramid schemes, both of which involve **geometric progressions** and **exponential growth**. Once you see where the mathematics is leading, you should appreciate why it is illegal to initiate chain letters or pyramid schemes, not only because they cause problems for the transmission system, but also because they usually involve fraud, potentially on a very large scale!

A Typical Chain Letter

Here is a typical illegal chain letter. If you ever get such a letter, be sure to ignore it.

> This is your lucky day! If you follow the instructions of this letter *exactly*, you will come into a huge windfall of money over the next several weeks. Below is a list of 7 names and addresses. Send 1 dollar to the first person on the list, then make a new list, deleting the top name, moving everyone else on the list up one place, and putting your name and address at #7. Copy this letter exactly, substituting the new list for the old list, and send it to six of your friends who can be trusted to not break the chain. In a few weeks you will receive a huge amount of money! You *must* participate for this to work!
> Do *not* break the chain or you will have a stroke of bad luck! A woman in Cleveland once broke the chain, and her TV set was hit by lightning right in the middle of her favorite TV show, *NUMB3RS*.
>
> (There would then follow a list of seven names and addresses.)

1. Suppose Notzo Smartt receives this letter and decides to participate. Of course, he has no way of knowing how many other letters are already out there. When he sends his letters out, how many lists will contain his name? ________________

2. Suppose none of Notzo's friends break the chain. If they follow instructions, their letters will contain a *second generation* of lists bearing Notzo's name. How many letters are in the second generation? ________________

3. Let us assume that all the letters in the second generation go to different people. Those people will create a third generation of lists bearing Notzo's name. How many letters are in the third generation? ________________

4. Explain why the assumption in Question 3 (that the second generation of letters will all go to different people) is probably unreasonable.

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5. Now let us assume that *all* the lists in *every* generation contain different names, and that nobody in any generation decides to break the chain. If every person on every list follows the instructions exactly, how many dollars will Notzo eventually receive?

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6. We noted in Question 1 that Notzo had no way of knowing how many letters were already out there. Suppose Notzo's name actually appeared in the *fourth* generation of letters after the original source. Again, assuming that no names repeat and that nobody breaks the chain, how many letters will be “out there” when Notzo's name finally gets to the top of his lists? _________________

7. Compare the number above to the current U.S. population, available at the Web site **http://www.census.gov/population/www/popclockus.html**. Do you see a possible problem for Notzo? ____

8. Who are the people most likely to profit from a chain letter? Who are the least likely? Can you see why chain letters are illegal? Explain.

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Geometric Progressions and Exponential Functions

The speed at which a chain letter grows depends on the number of friends to whom the letter must be sent.

9. Fill in the following table for a chain letter that includes a list of **ten** names, with instructions to send the letter to **five** friends. Your name takes ten generations to reach the top.

Generation	(Potential) Number of Letters with Your Name
1	5
2	
3	
4	
5	
6	
7	
8	
9	
10	

10. The numbers in the second column of the table above form a **geometric progression**. What simple rule is followed to get from one number to the next?

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11. Write the number in the second column y as a function of the number in the first column x. (This is an example of an **exponential function**.)

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12. In general, a geometric progression has the form $a, ar, ar^2, ar^3, \cdots$. An exponential function has equation $f(x) = a \cdot r^x$. What is the relationship between exponential functions and geometric progressions? [Hint: It has to do with *domain*.]

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Pyramid Schemes

In a *pyramid scheme* (another classic version of fraud), investors are coaxed into a "company" and are told to attract other investors. They earn back their investment through the new investors they attract, while paying the remainder to the "company" (that is, the investors on the levels above them). There may or may not be an actual product involved; everyone makes money as long as the company can attract new investors!

13. Explain mathematically why pyramid schemes and chain letters are doomed to failure for the same reasons.

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More Fun with Exponential Growth

- A typical sheet of newsprint has been folded over twice when it arrives in your daily newspaper. Try to fold it over eight more times. How many thicknesses of newsprint must you bend when you attempt the tenth fold (if you get that far)?
- Get a January calendar and put a penny on January 1. Stack two pennies on January 2, then four pennies on January 3, and eight pennies on January 4. That should be enough pennies to get the picture. Now take a calculator and figure out which day will have the first stack of pennies worth more than a million dollars. How tall (in miles) is the stack of pennies on January 31?

Challenge: Once you have the entire January calendar covered in pennies according to the rule above, what is the *total* number of pennies on the calendar? [Hint: The number of pennies on the *n*th day has a simple mathematical relationship to the total number of pennies accumulated *prior to* the *n*th day.] ________________

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The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Chain Letters

- An excellent description of chain letters and their various forms can be found at **http://www.cs.rutgers.edu/~watrous/chain-letters.html**.
- For a thorough history of chain letters and an impressive archive of them, see **http://www.silcom.com/~barnowl/chain-letter/evolution.html**.
- For a closer look at the kinds of chains that have plagued the Internet (some of which you will probably recognize), and for many more reasons not to forward these messages yourself, see **http://www.breakthechain.org**.

Pyramid Schemes

Pyramid schemes (also called **Ponzi Schemes**) prey on people who do not fully understand them, so the more you know about them the better. A nice description of them can be found at **http://members.impulse.net/~thebob/Pyramid.html**.

Folding Paper

In case you became frustrated trying to fold the newspaper and would like to see how thick it might have become if you *could* have kept folding it, you can discover the amazing truth by looking at **http://raju.varghese.org/articles/powers2.html**.

Stacking Pennies

If you enjoyed the exercise of stacking pennies on the calendar, maybe you would enjoy stacking pennies as a hobby. Don't laugh – or at least don't laugh until after you have checked out **http://www.fincher.org/CoinStacking/index.shtml**.

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