NUMB3RS Activity: Pile It On
Episode: "The Janus List"

Topic: Exponents and powers of two **Grade Level:** 9 - 10
Objective: Generalization of patterns involving exponents
Time: 15 - 20 minutes

Introduction

In "The Janus List," a bomber calls for Charlie and asks him a series of questions that involve a number of mathematical topics. One of the topics is the classic "Wheat and Chessboard" problem.

Discuss with Students

This activity is designed to enable students to explore exponential expressions and make generalizations. Students should observe that the number of grains of wheat on each square is a power of two. They may need to be taught (or reminded) that the number 1 can be expressed as 2^0.

Some students may observe that the number of grains of wheat on all 64 squares is the sum of a geometric sequence with a first term of 1 and a common ratio of 2.

In conjunction with Question 8, another way to appreciate just how large this amount of wheat is would be to imagine it in a pile. The *bushel* is the most common unit for measuring the volume of wheat. A bushel is 2150.42 cubic inches and would contain about one million grains of wheat. The pile in Question 7 would have a volume of about 156 cubic miles. A cone-shaped pile would have a diameter of about 14 miles and would be nearly 3 miles high. On the other hand, this pile would only fill about 6% of the Grand Canyon. Challenge students to compute the dimensions of the conical pile themselves using the following information:

- Compute the volume of the cone using the preceding information.
- The *angle of repose* – the slope of a natural pile of wheat – is 25°.
- The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

The Wheat and Chessboard problem and a puzzle called the Tower of Hanoi both involve calculating the sum of powers of two. Students can see this connection in the Extensions.

Student Page Answers:
1. *1, 2, 4, 8, 16, 32, 64, 128* **2.** $2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7$ **3.** *255* **4.** $256 = 2 \cdot 128$
5. $255 = 256 - 1 = 2^8 - 1 = 2 \cdot 2^7 - 1$. *The total number of grains of wheat on the first eight squares is one less the number of grains on the ninth square.* **6.** 2^{63} **7.** $2 \cdot 2^{63} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$
8. *Between 7,000 and 8,000 years at the current rate of production.* **9.** *Because tic-tac-toe has nine squares, the Grand Vizier would have received 511 grains of wheat. He would have been better off getting gold instead.*

Extension Page Answers:
1. *The pattern, except for the first square, is the repeating pattern 2, 4, 8, 6, 2. Since the last digit of the 4th square is 8, then the last digit of the 64th square is also 8.* **2.** *There would be* 3^{63} *grains of wheat on the 64th square and* $(3^{64} - 1) / 2$ *total grains on the chessboard.* **3.** *59 minutes, about 53 minutes, about 43 minutes,* 2^{60} *or about 1.15 quintillion liters.* **4a.** *15 moves are required.* **4b.** *Quarter: 1 move; nickel: 2 moves; penny: 4 moves; dime: 8 moves.* **4c.** *The tables have the same numbers in the same order,* $2 \cdot 2^{63} - 1$ *moves.*

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Name: ________________________________ Date: _______________

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In “The Janus List,” a bomber calls for Charlie and asks him a series of questions that involve a number of mathematical topics. One of the topics is the classic “Wheat and Chessboard” problem. Some credit this problem to an Indian myth. According to the myth, chess was invented by Grand Vizier Sissa Ben Dahir, who gave the game to King Shirham of India. The king offered a reward of gold, but the Grand Vizier stated that he would prefer to just have some wheat: one grain for the first square of the chessboard, two grains for the second square, four for the third, and so on, doubling each time. The king granted this apparently modest request. The problem is to determine how many grains of wheat are required to cover the 64 squares of the chessboard in this manner. The problem was first published by the Arabic mathematician Ibn Kallikan in 1256.

1. Fill in the first row of the chessboard with the number of grains of wheat for each square.

2. Rewrite your answers to Question 1 using exponential notation.

3. What is the total of the number of grains of wheat on these eight squares? What do you think of the deal the Grand Vizier made with the King?

4. How many grains of wheat will be placed on the ninth square? How does this relate to the number of grains of wheat on the eighth square?

5. What is the relationship between your answers to Questions 3 and 4?

6. Write an exponential expression for the number of grains of wheat on the 64th square.

7. Use the answers to Questions 5 and 6 to solve the original problem. Write an expression for the total number of grains of wheat on all 64 squares.

8. According to the Kansas Wheat Commission, there are roughly a million grains of wheat in a bushel. The total U.S. wheat production is 2.4 billion bushels per year. How long would it take American farmers to grow enough wheat to compensate the Grand Vizier?

9. Suppose the Grand Vizier had invented tic-tac-toe instead of chess. Should he have made his request to the King to take his reward in wheat in a similar manner?

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

For the Student

1. Consider the digits in the ones places of the numbers in the answer to Question 1 (except for 1). Predict the digits in the ones place of the number of grains of wheat for the next eight squares of the chessboard. Verify this prediction with your calculator. What is the digit in the ones place of the number of grains of wheat on the 64th square?

2. Suppose that the number of grains of wheat were to triple each time rather than double. Answer Questions 1 – 7 with this assumption and determine the total number of grains of wheat on the chessboard.

3. Suppose a certain kind of bacteria doubles in volume every minute. A small quantity of these bacteria was placed in a one-liter jar. After one hour (60 minutes) the jar was full. When was it half full? When was it 1% full? Suppose the bacteria cannot be seen with the naked eye until the jar is 0.001% filled. When did it first become visible? Suppose that the top is left off of the jar. What will the volume of bacteria be after another hour?

4. The Tower of Hanoi is another classic mathematical puzzle. One version uses four coins: a dime, a penny, a nickel, and a quarter, stacked in that order on an index card, plus two other index cards. The object is to move the stack from one index card to another, using the extra card if necessary, one coin at a time, with the provision that only one coin moves at a time and a larger coin can never be placed on top of a smaller coin. The problem is to determine the minimum number of moves required to accomplish this task.

a. Use either four different coins or four different-sized squares of paper to solve the Tower of Hanoi puzzle. How many moves are required?

b. Move the stack again, according to the Tower of Hanoi rules. This time, record how many times you move each coin.

quarter	**nickel**	**penny**	**dime**

c. Compare the answer above to the answers to Questions 1 and 2 in the activity. What do you observe? The Tower of Hanoi puzzle can be done with any number of disks. How many moves should it take to move a tower of 64 different diameter disks?

Additional Resource

The Web site below uses salt instead of wheat for the chessboard problem and includes interesting comments and observations about the magnitude of each of the 64 numbers.
http://www.averypickford.com/third/salt.htm

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