Math Objectives

- Students will find roots to equations using the recursive algorithm Fixed Point Iteration (FPI) and will visualize the algorithm as a cobweb plot.

- Students will use appropriate tools strategically (CCSS Mathematical Practice).

Vocabulary

- recursive sequence
- convergent sequence
- Fixed Point Iteration
- cobweb plot

About the Lesson

- This lesson involves finding roots to equations using a method similar to those used by many calculators.
  As a result, students will:
  - Find roots using a recursive sequence - Fixed Point Iteration.
  - Visualize this root-finding recursive sequence as a cobweb plot.

TI-Nspire™ Navigator™ System

- Transfer a File.
- Use Screen Capture to monitor student progress.
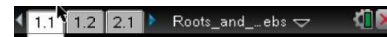
Roots and Cobwebs

To find a root to the equation F(x) = 0, a calculator often uses a **recursive (iterative) algorithm**. In this activity you will study one such algorithm and its visual representation as a **cobweb plot.**

TI-Nspire™ Technology Skills:
- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:
- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing [ctrl] [G].

Lesson Files:
Student Activity
Roots_and_Cobwebs_Student.pdf
Roots_and_Cobwebs_Student.doc

TI-Nspire document
Roots_and_Cobwebs.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

To use FPI to determine a sequence $\{x_n\}$ that converges to a root of F(x) = 0:

1) Rewrite $F(x) = 0$ in the form $x = f(x)$ [there are often many choices for $f(x)$.]

2) Choose an initial guess for a solution, x_0

3) Calculate $x_n = f(x_{n-1})$ n = 1, 2, .. until $\{x_n\}$ converges, i.e. $|x_n - x_{n-1}|$ is very small.

If $\{x_n\}$ does not converge, return to step 1, and make another choice for $f(x)$.

Problem 1: Estimate the root of $\sqrt{x} - x + 1 = 0$ using FPI.

According to Step (1) above, we must rearrange $F(x) = \sqrt{x} - x + 1 = 0$ into the form $x = f(x)$. One method to find a function $f(x)$ is: $\sqrt{x} - x + 1 = 0 \rightarrow x = \sqrt{x} + 1$ so that $f(x) = \sqrt{x} + 1$. For Step (2), we choose $x_0 = 4.0$ as an initial guess for a root.

Move to page 1.2.

For Step (3), we can find the values $x_i, i = 1, 2, 3, \ldots$ using a spreadsheet. This process is illustrated on Page 1.2 for the initial guess $x_0 = 4.0$, using the following procedure:

*   Type a title for the first column (**root1**).
*   Type 4.0 into cell A1, and press [enter].
*   In cell A2, type $= \sqrt{a1} + 1$ [$= f(a_1)$], and press [enter].
*   Arrow up (▲) to select cell A2.
*   Select **MENU > Data > Fill**.
*   Press the down arrow (▼) until you reach cell A20, and press [enter] to fill in the cells.

The column of values suggests that the sequence converges to 2.61803.

> **Teacher Tip:** Remind students to enter an initial guess as 4.0 [rather than 4] so the values in the spreadsheet are in decimal form.

1. Using Scratchpad, verify that 2.61803 is a root of $\sqrt{x} - x + 1 = 0$.

Answer: $\sqrt{2.61803} - 2.61803 + 1 = .000003$. Because 2.61803 is an approximation, this value is not exactly 0.

2. a. Using the spreadsheet, find whether the FPI sequence determined by $f(x) = \sqrt{x} + 1$, with each given number below as the initial guess, converges to the root 2.61803.

    i) 1.0   ii) 8.0   iii) 12.0

    Answer: In all three cases, the FPI sequence converges to the root.

    b. For what values of x_0 do you think the resulting FPI sequence will converge to the root 2.61803?

    Answer: Trying large values for x_0 such as x_0 = 1000 suggests that the sequence will converge for $x_0 > 0$

Note: To change the initial term in the sequence, change the value of cell A1 to the initial guess. The remaining values in the column corresponding to this initial guess are automatically updated.

3. Let's try another choice for $f(x)$, namely $f(x) = (x-1)^2$. Show how $\sqrt{x} - x + 1 = 0$ can be rewritten in the form $x = f(x)$ to determine this choice of $f(x)$.

    Answer: $\sqrt{x} - x + 1 = 0 \rightarrow \sqrt{x} = x - 1 \rightarrow x = (x-1)^2$

4. Does the FPI sequence using this choice for $f(x)$ with initial guess $x_0 = 4.0$ seem to converge to the root 2.61803? Use Column B on the spreadsheet and the procedure above to populate the column and answer the question.

    Answer: No. The values in the sequence increase very rapidly and quickly approach ∞.

5. Does the FPI sequence determined by $f(x) = (x-1)^2$ converge to the root 2.61803 for any choice(s) of the initial guess x_0?

    Answer: No. For $0 \le x_0 \le root \approx 2.61803$, the values in the sequence eventually oscillate between 0 and 1; for $x_0 > root \approx 2.61803$, the values in the sequence increase very rapidly and quickly approach ∞.

Hint: Try various choices for the initial guess, x_0, in the second column of the spreadsheet used in Question 4.

> **TI-Nspire Navigator Opportunity:** *Screen Capture*
> **See Note 1 at the end of this lesson.**

Problem 2a: This problem illustrates the ancient Babylonian algorithm for estimating a square root. Perhaps the first algorithm used to approximate $\sqrt{a}$ is the "Babylonian method", or "Heron's method", named after the first-century Greek mathematician Heron of Alexandria who gave the first explicit description of the method. Heron's method is, perhaps, the first case where FPI was used historically. Here is an illustration of this method for approximating $\sqrt{5}$:

We can rewrite $F(x) = x^2 - 5 = 0$ as $x = \dfrac{5}{x}$ → $2x = x + \dfrac{5}{x}$ → $x = \dfrac{\left(x + \dfrac{5}{x}\right)}{2}$, so that

$f(x) = \dfrac{\left(x + \dfrac{5}{x}\right)}{2}$ is one choice for $f(x)$.

> The logic behind this choice for $f(x)$ is that if x is a good guess for $\sqrt{5}$, then so is $\dfrac{5}{x}$ since
>
> $x \cdot \dfrac{5}{x} = 5$. Their average $\dfrac{\left(x + \dfrac{5}{x}\right)}{2}$, then, should be a better guess.

Move to page 2.1.

One method of implementing FPI is to find $\sqrt{5}$ through iteration on a spreadsheet as in Problem 1. This process is illustrated for the initial guess $x_0 = 4.0$ on Page 2.1 by using the following

procedure:

*   Type a title for the first column ($sqr5$).

*   Type 4.0 into cell A1, and press [enter].
*   In cell A2, type: $= (a1 + 5 / a1) / 2$ [$= f(a_1)$], and press
    [enter].

*   Arrow up (▲) to select cell A2.

*   Select **MENU > Data > Fill**.

*   Press the down arrow (▼) until you reach cell A20, and
    press [enter] to fill the cells.

The sequence of values seems to approach 2.23607 very rapidly since the fifth and sixth iterations were both equal to 2.23607.

6. a. Using the spreadsheet, find the fifth number in the FPI sequence determined by

$f(x) = \dfrac{\left(x + \dfrac{5}{x}\right)}{2}$, using each given number below as the initial guess.

i) 6.0   ii) 1.0   iii) -8.0   iv) 10.0

Answer: a) i) 2.23608   ii) 2.23607   iii) -2.23652   iv) 2.23916

   b. In each case, does the sequence seem to converge to $\sqrt{5}$? If not, to what does it seem to converge?

Answer: i), ii), and iv) seem to converge to $\sqrt{5}$ while iii) seems to converge to $-\sqrt{5}$ since the sequence tends to -2.23607.

Note: To change the initial term in the sequence, change the value of A1 to the initial guess. The remainder of the values corresponding to this initial guess are automatically updated.

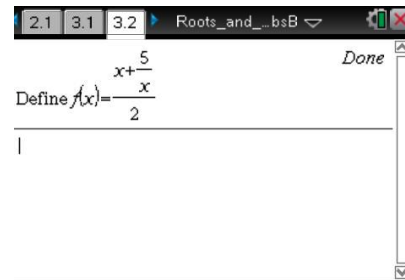**Move to page 3.1.**

Problem 2b: Cobweb Sequence Plotter
- Define the iterative function $f(x)$ on Page 3.2.
- On Page 3.3, the graphs of $y = f2(x) = f(x)$ and $y = f1(x) = x$ are shown.
- The k -clicker steps through the iterations of f composed with itself.
- The graph shows a starting point $(x_0, f(x_0))$ and the intermediate "caroms" to the line $y = x$.
- The x -clicker can be used to change the initial value of x (or, alternatively, you can just drag the point on the x-axis to change the initial value).

COBWEB SEQUENCE PLOTTER

INSTRUCTIONS: Define the iterative function f(x) on Page 3.2

On Page 3.3, the graphs y = f1(x) = x and

y = f2(x) = f(x) are shown.

The k–clicker steps through the iterations

of f composed with itself. The graph shows a

starting point (x, f(x)) and the intermediate
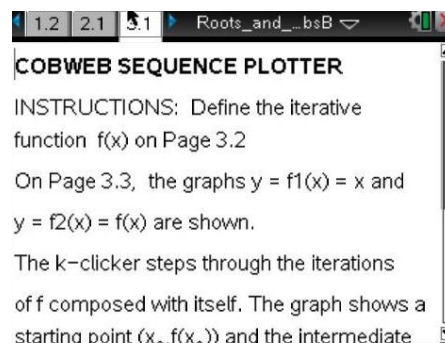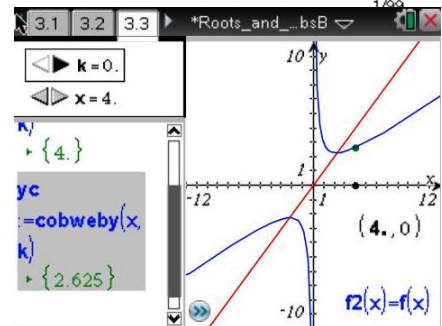
Move to page 3.2

The function $f(x) = \dfrac{\left(x + \dfrac{5}{x}\right)}{2}$ is defined on page 3.2.

Move to page 3.3.

The initial value of $x_0 = 4.0$ has been selected with the x-clicker.

7. Using the k-clicker, cycle through the values from $k = 0$ to $k = 7$.
    a. Explain how the segments going back and forth between $y = f(x)$ and $y = x$ in the graph illustrate a sequence of values tending to an estimate of $\sqrt{5}$.

    Sample Answer: These two steps are applied repeatedly:
    (1) moving from $y = x$ [or the x-axis] to the curve illustrates the calculation of $f(x)$; (2) moving from the curve to $y = x$ illustrate setting the value of $f(x)$ as the next *x*-value in the sequence.

    b. How do the values in the lower left of the screen compare to those in the spreadsheet?

    Sample Answer: Each value in the spreadsheet occurs twice in the cobweb plot – once as an *x*-value and once as an $f(x)$ -value.

> **Teacher Tip:** To make sure that the current values of the cobweb sequences are visible in the window at the lower left, remind students to move it as far to the right as possible by clicking on the arrow in the middle-right of the window and to arrow up or down as needed.

8. Use the x -clicker to select each of the values of x_0 listed below, and then use the k -clicker to cycle though the values $k = 0$ to $k = 7$.
    i) 6.0   ii) 1.0   iii) -8.0   iv) 10.0
    a. In each case, does the cobweb graph illustrate the calculation of an estimate of $\sqrt{5}$?
    b. If not, what does it illustrate, if anything?

    Answer: The sequence with an initial value of 6.0, 1.0, or 10.0 seems to converge to $\sqrt{5}$; the sequence with initial value -8.0 seems to converge to - $\sqrt{5}$.

Note: You need to reset the k-clicker to 0 before considering a new value of x_0. You might have to type the initial value x_0 into the x-clicker to cycle through integer values of x.

TI-Nspire Navigator Opportunity: *Live Presenter*
See Note 1 at the end of this lesson.

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9. Leave the graph corresponding to $x_0 = 10$ displayed. By dragging the initial value along the x-axis, determine the values of the initial guess x_0 for which the FPI sequence seems to converge to $\sqrt{5}$ (i.e., the sequence of segments converges to the point of intersection of the graphs $y = f1(x) = x$ and $y = f2(x) = f(x)$ in the first quadrant).

What happens for the other values of x between -10 and 10 for which the sequence does not seem to converge to $\sqrt{5}$?

Answer: The FPI sequence seems to converges to $\sqrt{5}$ when the initial guess $x_0 > 0$; the sequence is undefined when $x_0 = 0$, and the sequence seems to converges to $-\sqrt{5}$ when $x_0 < 0$.

Problem 3: The equation $x^3 + x - 6 = 0$ has a root between 1 and 2. Estimate this root using FPI.

Consider these four choices for $f(x)$.

$\quad$ i) $\sqrt[3]{6-x}$ $\qquad$ ii) $6 - x^3$ $\qquad$ iii) $\dfrac{6}{x^2+1}$ $\qquad$ iv) $\sqrt{\dfrac{6-x}{x}}$

10. Show how $x^3 + x - 6 = 0$ can be rewritten in the form $x = f(x)$ for choices iii) and iv).

Sample Answer: iii) $x^3 + x - 6 = 0 \rightarrow x(x^2 + 1) = 6 \rightarrow x = \dfrac{6}{x^2+1}$

iv) $x^3 + x - 6 = 0 \rightarrow x^2 = \dfrac{6}{x} - 1 = \dfrac{6-x}{x} \rightarrow x = \sqrt{\dfrac{6-x}{x}}$

Move back to page 2.1.

As we have seen in Problem 1, not all choices of $f(x)$ lead to a sequence converging to a root and some of the sequences that do converge to a root do so very slowly.

11. For each of these four choices of $f(x)$, use the process below, starting with an initial guess x_0, to determine whether the FPI sequence of values converge to the root. If yes, what is the root? If not, what happens to the sequence?

    Sample Answer: a) The sequence converges to 1.63437. b) The values in the sequence increase very rapidly [both positively and negatively] so that they quickly oscillate between $-\infty$ and ∞. c) The sequence oscillates between the two values .171573 and 5.82843. d) The sequence converges to 1.63437 very slowly.

    - Type a title (root1, for example) for the next column (Column B, for example).
    - Type your initial guess x_0 (2.0 for example) into cell B1 (for example), and press ⌷enter⌷.
    - In cell B2, type: $= f(b1)$ (For $f(x) = \sqrt[3]{6-x}$, type: $= (6-b1)\wedge(1/3)$ and press ⌷enter⌷.
    - Arrow up (▲) to select cell B2.
    - Select **MENU > Data > Fill**.
    - Press the down arrow (▼) until you reach cell B20, and press ⌷enter⌷ to fill the cells.

Move back to pages 3.2 and 3.3.

Verify your results for Question 11 by creating the cobweb plot for each of the choices of $f(x)$ by first typing $define\ f(x) =$ (new choice for $f(x)$) on Page 3.2 and then examining the cobweb plots on Page 3.3.

12. Describe how the cobweb plot for each choice of $f(x)$ supports your answer for Question 11.

    Sample Answer: a) The segments going back and forth between $y = f(x)$ and $y = x$ converge to 1.63437. b) The segments going back and forth between $y = f(x)$ and $y = x$ quickly go outside the window—the values in the box in the lower left increase to large positive and negative values very quickly. c) The segments going back and forth between $y = f(x)$ and $y = x$ form a series of nested squares where the values "bounce back and forth" between .171 and 5.82. d) Starting from an initial guess of $x_0 = 2$, the segments going back and forth between $y = f(x)$ and $y = x$ "clump around" the root 1.63437.

> **TI-Nspire Navigator Opportunity:** *Screen Capture*
> **See Note 1 at the end of this lesson.**

13. For each choice of $f(x)$ for which its FPI sequence converges to the root, use the process from Question 9 of dragging the initial value along the x-axis to determine the range of possible values of x_0.

Sample Answer: $\sqrt[3]{6-x}$: converges for $-10 < x_0 < 10$—the limits of the window; it actually converges for all real values of x_0. $\sqrt{\dfrac{6-x}{x}}$: converges for $0 < x_0 < 6$.

Extension

Find the 3 roots of the cubic polynomial $x^3 - 3x + 1 = 0$ using FPI. Determine a suitable choice of $f(x)$ for each root, and verify that the FPI sequence of values approaches the root using both the spreadsheet and the associated cobweb plot.

Sample Answer: a) .347 using $f(x) = \dfrac{x^3+1}{3}$ or $-\dfrac{1}{x^2-3}$ [from $x(x^2-3)+1=0$];

b) 1.53 using $f(x) = \sqrt[3]{3x-1}$;

c) -1.88 using $f(x) = -\sqrt{\dfrac{3x-1}{x}}$ [from $x^3 = 3x-1 \rightarrow x^2 = 3 - \dfrac{1}{x} = \dfrac{3x-1}{x}$].

Hint: You will need a different function, $f(x)$, for each root.

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:
*   Use the FPI recursive sequence to find roots of a polynomial.
*   Understand how a cobweb plot is a visual representation of the FPI algorithm.

TI-Nspire Navigator

Note 1 Name of Feature: Screen Capture

Use Screen Capture to monitor student use of the spreadsheet including choice of initial guesses.

Note 2 Name of Feature: Live Presenter

Use Live Presenter to illustrate and explain a cobweb plot.

Note 3 Name of Feature: Screen Capture

Use Screen Capture to monitor student use of the cobweb plot and defining the functions $f(x)$ including remembering to use $del\, \mathrm{var}\, f$ to delete one function before defining another.