Understanding the Derivative Backward and Forward

by Dave Slomer

Slopes of lines are important, giving average rates of change. Slopes of curves are even more important, giving *instantaneous* rates of change. This activity will define procedures for computing instantaneous rates of change exactly and, when exact is impossible or impractical, approximately. To do this requires a rigorous definition of *slope of a curve*.

The slope of a curve at a point is defined to be the slope of the line *tangent* to the curve at that point (see figure 1a). But the slope formula requires two points to get a slope. Often, as in figure 1a, a tangent line passes through only one point on a curve. So where do we get another point from which to compute the slope? We look to any other "nearby" point. [When they exist, tangent lines can be understood intuitively by zooming in at the point of tangency and noticing that, after enough zooms, the tangent and the curve appear to be the same graph. Hence, the curve, being virtually linear in that window, can be thought of as having a definite slope. Before proceeding, exploring this phenomenon via another activity might be advisable.]

Fig. 1a

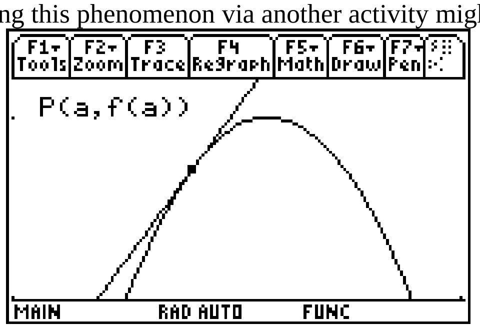
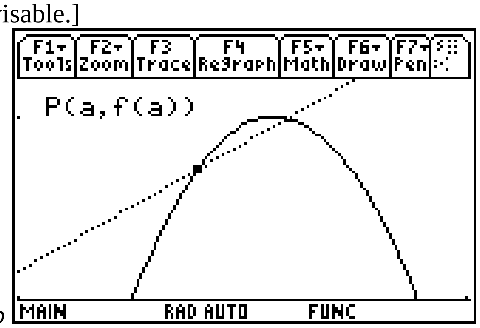


Fig. 1b



If f is any function, then $\dfrac{f(x)-f(a)}{x-a}$ is the slope of the *secant* line (see figure 1b) through $(a, f(a))$ and $(x, f(x))$. Because it is a quotient in which both the numerator and denominator are differences, that expression is called a ***difference quotient***. Any secant line goes through (at least) two points on the graph of a function; hence the numerator will always be the difference of two function values. If we want to somehow use the secant line to approximate the slope of the tangent, then x should be "near" a.

Exercise 1: The point $(x, f(x))$ is referred to in the numerator of the difference quotient above. In figure 1b, where is that point? Is that point near enough to P to give a decent approximation of the slope of the tangent at P (shown in fig. 1a)?

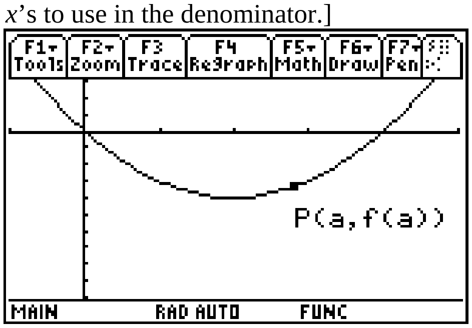
Other forms of difference quotients include $\dfrac{\Delta f}{\Delta x}$ (which is very generic, but concise) and $\dfrac{f(a+\Delta x)-f(a)}{(a+\Delta x)-a}$. If the latter two difference quotients are going to give slopes that are close to the slope of the tangent at $(a, f(a))$, their denominators are going to have to be near 0. (Convince yourself of that.)

Exercise 2: Simplify the difference quotient $\dfrac{f(a+\Delta x)-f(a)}{(a+\Delta x)-a}$. (After simplification, does it still look like a difference of two function values over the difference of two x's?)

- For this difference quotient to be a good approximation of the slope of the tangent line at $(a, f(a))$, Δx must be near ______.

If Δx is positive, the point $(a + \Delta x, f(a + \Delta x))$ will be to the right of $(a, f(a))$, so the difference quotient in Exercise 2 is called the ***forward difference quotient.***

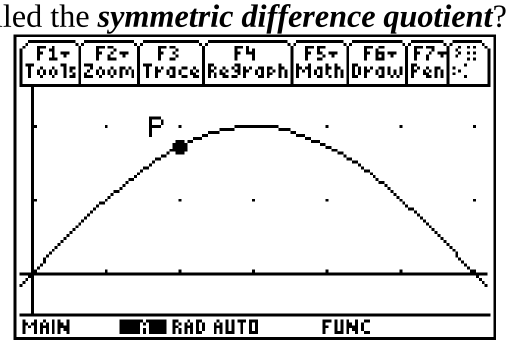
Exercise 3: The difference quotient $\dfrac{f(a)-f(a-h)}{h}$ is the slope of the secant line through two points on the graph of *f*. Assume that *h* is positive and locate the other point on the sketch below and draw the secant line. [The function expressions in the numerator tell you what x's to use in the denominator.]



- What are the coordinates of the two points represented in the difference quotient above?
- In order for this secant line to be a good approximation to the slope of the tangent line at *P*, then *h* would have to be near _____.

If *h* is positive, the point $(a – h, f(a – h))$ will be to the left of $(a, f(a))$, so the difference quotient in Exercise 3 is called the ***backward difference quotient.***

Exercise 4: On the graph below, each tick mark represents one unit and *P* is the point $(a, f(a))$. Let *h* be 0.5, which is not particularly close to 0, and draw the secant line through the points $(a – h, f(a – h))$ and $(a + h, f(a + h))$. Then express its slope as a difference quotient in terms of *f*, *a*, and *h*. Do you see why this difference quotient is called the ***symmetric difference quotient***?



- Using the grid dots on the graph screen, estimate the slope given by the symmetric difference quotient.
- Draw the tangent line at *P*. How well does the symmetric difference quotient approximate its slope?

Exercise 5: The function in the graph in Exercise 4 is $f(x) = \sin\dfrac{\pi x}{3}$ and *P* has *x*-coordinate 1.

- Graph that function, zoom in [with equal zoom factors] at *P* until the graph looks linear, and estimate the slope.
- Compute the symmetric difference quotient with $h = 0.5$.
- For what value of *h* are the forward and backward difference quotients as accurate as the symmetric with $h = 1$?
- Conclusions?

Exercise 6: Show algebraically [feel free to use your '89] that the symmetric difference quotient is the average of the forward and backward difference quotients. (For aid, refer to figure 2, in which the forward difference quotient gives the slope of $\overleftrightarrow{PR}$, while the backward gives the slope of $\overleftrightarrow{PQ}$, and the symmetric gives the slope of $\overleftrightarrow{QR}$. You would be wise to let $(a, f(a))$ represent *P* and use either *h* or Δx (consider both to be positive) to determine the coordinates of *Q* and *R*.)

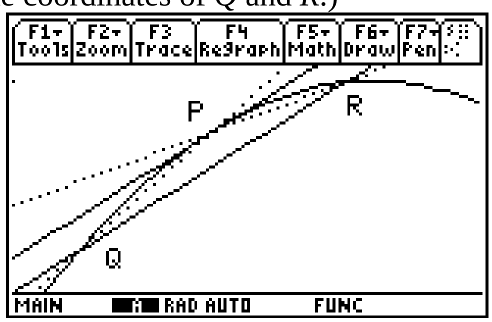
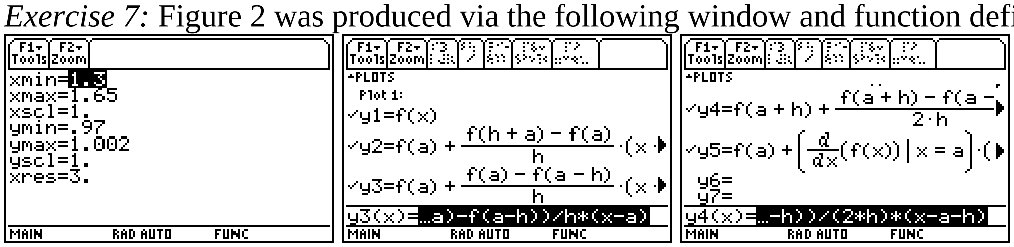


Fig. 2

Exercise 7: Figure 2 was produced via the following window and function definitions.



- Match the tangent line and forward, backward, and symmetric difference quotient secant lines with the appropriate function in **y2** through **y5**.
- Each equation in **y2** through **y5** is a line in point-slope form, but every function defintion is at least partly cut off at the right margin. Write the equations for **y2** through **y5**.

[FYI—figure 3 below was produced by turning off **y3** through **y5** and storing {.1,.075,.05,.25} into **h**.]

OK, fine. We can find a lot of different types of secant lines whose slope can be expressed as many different-looking difference quotients that are related in an interesting way (Exercise 6). But how do we find slopes of <u>tangents</u>?

Refer to figure 3 to see several "forward" secant lines (for h = .1, .075, .05, .025) that approach, as a limit, the tangent line, which is also included in figure 3.

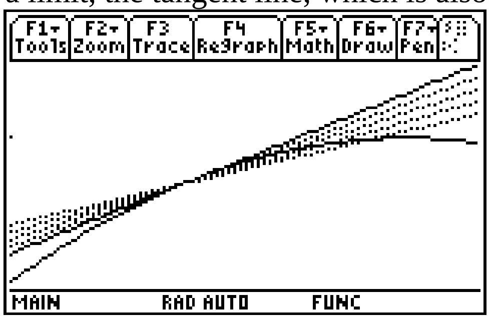


Fig. 3

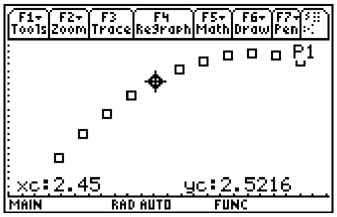
Exercise 8: In figure 3, where is the point of tangency? Which line is the tangent? If the point of tangency is $(a, f(a))$, what would be the equation of the tangent there?

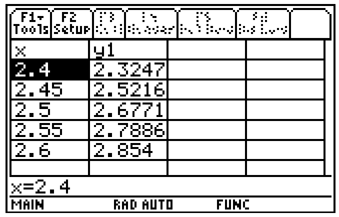
It looks like the slope of the tangent line in figure 3 is a positive number. When it exists [sometimes it doesn't], the slope of the line tangent to the function $y = f(x)$ at $(a, f(a))$ is defined as $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$. This is called the ***derivative of f at a***.

Exercise 9: Which difference quotient is used in the definition of derivative? Could either or both of the other two also have been used? Explain.

If, for some reason, the limits cannot be computed, the derivative can be approximated by using any of the individual difference quotients, if a small enough h can be used. As figure 2 shows, the backward and forward difference quotients are not likely to very good at estimating the slope of the tangent line unless h is very close to 0. But, as figure 2 also shows (at least for the function involved), the symmetric difference quotient can be very close to correct, even if h is not so small, or if we have only a graph to proceed from.

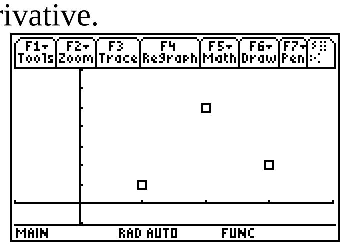
Exercise 10: For the function defined graphically and numerically below, estimate its derivative at x = 2.45 using all 3 difference quotients.

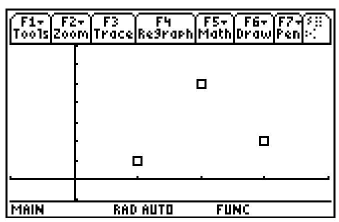


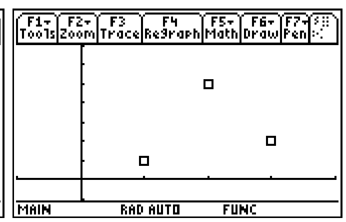


- The data above came from the function $f(x) = 2\sin(3x) - \cos(x)$. Use the definition of derivative (a limit of a forward difference quotient) to calculate the exact value of the derivative.
- Your TI-89 can calculate <u>many</u> derivatives exactly. **Define f(x)=2sin(3x)–cos(x)** and then enter the command ***d*(f(x),x)|x=2.45**. [***d*** is above the 8 key.] Did you get the same result that you did in the first bullet above? (Just checking!)
- Which difference quotient came closest? By how much did it miss? What is the relative error in the calculation?
  [Relative error is defined as |actual value – approximate value|/actual value.]
- Which missed by most? By how much? What is the relative error?
- Conclusions?

Exercise 11: Consider the points (5,5), (10,25), and (15,10), plotted on 3 axes below. The fact is that any one of the 3 difference quotients could give the best estimate of the derivative.







- Draw a function for which the FDQ would be the best estimate of the derivative at (10,25).
- Draw a function for which the BDQ would be the best estimate of the derivative at (10,25).
- Draw a function for which the SDQ would be the best estimate of the derivative at (10,25).
- Which of the 3 functions you graphed do you think would be most likely to occur in a “real-life” problem? Why?

Exercise 12: The symmetric difference quotient is so good that it gives <u>exact</u> results for parabolas, even without taking a limit! Prove that this is so.

Exercise 13: Unlike the TI-89, some calculators, such as the TI-83, cannot compute limits, since they cannot do symbol manipulation. Still, they have the built-in capability to find derivatives approximately. Why do you think they use the symmetric difference quotient to numerically approximate derivatives?

Exercise 14: Explain why the symmetric difference quotient (and therefore the TI-83, for example) gives 0 as an approximate value of the derivative of $|x|$ at 0. What do you get if you take the limit of the symmetric difference quotient for $|x|$ at 0? (Hint: take the left- and right-hand limits when you compute the derivative from the definition.) Conclusions?

Since the derivative is a limit, x can approach a from both the right ($x > a$; see figure 1b) and the left ($x < a$), so it makes sense to then talk about ***left-*** and ***right-hand derivatives***. If $f'(a)$ exists, the left- and right-hand derivatives at a are equal, and all 3 derivatives equal the slope of the tangent line at $(a, f(a))$. (See figure 1.)

Sometimes x cannot approach a from both sides. If $f(x) = \left(\sqrt{-x}\right)^4 + x$, the derivative only exists at 0 from the left. (Why?) Other times, the function may not even have a derivative, having different left- and right-hand derivatives. (Can you think of an example or two of such a function?) And even if a function has a derivative for all x, if the function is piecewise-defined, it would be necessary to compute the left- and right-hand derivatives separately at any point where the function definition changes from one piece to the next.

Thus, the following definitions are necessary:
The ***right-hand derivative*** of f at a, often designated as $f'(a)^+$, is defined as $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$. Similarly, the ***left-hand derivative***, $f'(a)^-$, can be defined by either $\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$ or $\lim_{h\to 0^+} \frac{f(a)-f(a-h)}{h}$.

Exercise 15: Compute the left-hand derivative for $f(x) = \left(\sqrt{-x}\right)^4 + x$ at 0.

Exercise 16: For $f(x) = \begin{Bmatrix} \ln(x), 0 < x < 1 \\ (x-2)^2 - 1, x \geq 1 \end{Bmatrix}$, find the left- and right-hand derivatives at 1 and let them tell you whether the derivative of f exists at 1.

- Draw the graph of the derivative. Describe its behavior at 1. Conclusions?

Calculus Generic Scope and Sequence Topics: Derivatives
NCTM Standards: Number and operations, Algebra, Geometry, Measurement, Connections, Communication, Representation