NUMB3RS Activity: Right or Wrong
Episode: "Mind Games"

Topic: The Binomial Theorem **Grade Level:** 9 - 12
Objective: Find binomial probabilities and interpret their meaning
Time: 15 - 20 minutes
Materials: TI-83/84 Plus calculator

Introduction

In "Mind Games," a psychic shocks Charlie by incorrectly guessing the color (black or red) of 25 cards in a row. As Larry says, "the probability of getting them all wrong is the same as getting them all right." This activity applies the binomial theorem to the probability of correctly guessing a particular outcome for a number of events. The activity is written to apply to various approaches and grade levels. Students must know how to expand $(R + W)^n$ for $n \leq 4$. If your students have studied Pascal's triangle, this activity will provide a nice connection. If your students understand the more formal Binomial Theorem, $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$, it can also be used in the activity. For these students, some of the extensions might appropriately be included as part of the activity.

Discuss with Students

Students should be familiar with the definition of simple probability as well as the fact that the probability of two or more independent events is found by multiplying the individual probabilities. In addition, they should be aware that the probability of the complement of a particular event is (1 – the probability of the event). In this activity we consider R (probability of guessing right) and W (probability of guessing wrong) to be complementary, so that $R + W = 1$. Thus, $(R + W)^n$ is also equal to 1. Additionally, n will represent the number of events (guessing cards or answers to multiple choice questions).

On the student page, the notation ${}_nC_r$ is used for combinations because it matches the notation on your calculator. If your students are more familiar with the $\binom{n}{r}$ notation, remind them that these notations are equivalent.

You may also want to review the basics of combinations with your students. In general, the number of ways of selecting from n objects taken r at a time, without regard to the order in which one selects them, is written ${}_nC_r$. This is defined as $\frac{n!}{r!(n-r)!}$. (Remember that $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$, and $0! = 1$).

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Student Page Answers:

1. *$R^3 = \frac{1}{8}$ = 0.125 (The probability of guessing all three right is $\frac{1}{8}$); $3R^2W = \frac{3}{8}$ = 0.375 (The probability of guessing two right and one wrong is $\frac{3}{8}$); $3RW^2 = \frac{3}{8}$ = 0.375 (The probability of guessing one right and two wrong is $\frac{3}{8}$); $W^3 = \frac{1}{8}$ = 0.125 (The probability of guessing all three wrong is $\frac{1}{8}$)*

2. *$R^4 + 4R^3W + 6R^2W^2 + 4RW^3 + W^4$; both are $\frac{1}{16} = 0.0625$* **3.** *$\frac{6}{16} = 0.375$*

4.

Expression	Expansion	Probabilities
$(R + W)$	$R + W$	$\frac{1}{2} + \frac{1}{2}$
$(R + W)^2$	$R^2 + 2RW + W^2$	$\frac{1}{4} + \frac{2}{4} + \frac{1}{4}$
$(R + W)^3$	$R^3 + 3R^2W + 3RW^2 + W^3$	$\frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$
$(R + W)^4$	$R^4 + 4R^3W + 6R^2W^2 + 4RW^3 + W^4$	$\frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16}$

5.

Number of Cards	Probability of all correct	Probability of *exactly half* correct	Probability of all wrong
4	$\frac{1}{16} = 0.0625$	$\frac{6}{16} = 0.375$	$\frac{1}{16} = 0.0625$
6	$\frac{1}{2^6} = 0.015625$	$\frac{20}{2^6} = 0.3125$	$\frac{1}{2^6} = 0.015625$
10	$\frac{1}{2^{10}} = 0.0009765625$	$\frac{252}{2^{10}} = 0.24609375$	$\frac{1}{2^{10}} = 0.0009765625$
20	$\frac{1}{2^{20}} \approx 0.0000009536743$	$\frac{184{,}756}{2^{20}} \approx 0.176197052$	$\frac{1}{2^{20}} \approx 0.0000009536743$
30	$\frac{1}{2^{30}} \approx 0.0000000009313$	$\frac{155{,}117{,}520}{2^{30}} \approx 0.144464448$	$\frac{1}{2^{30}} \approx 0.0000000009313$

6. *No, it is only one-half for 2 cards. It appears to decrease as the number of cards increases.*

7. $\left(\frac{1}{4}\right)^{25} \approx 8.88178 \times 10^{-16}$

8. $\left(\frac{3}{4}\right)^{25} \approx 7.52543 \times 10^{-4}$

9. $({}_{25}C_{20})\left(\frac{1}{4}\right)^{20}\left(\frac{3}{4}\right)^{5} \approx 1.14669 \times 10^{-8}$

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Name: ________________________________ Date: ________________

NUMB3RS Activity: Right or Wrong

In "Mind Games," Charlie is surprised when a psychic attempts to predict the color of 25 cards and gets every card wrong. Because it is equally likely that each card is red or black, the probability of correctly (or incorrectly) guessing the color of each card is $\frac{1}{2}$.

For two cards, the probability of guessing the right color for both cards is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ or $\frac{1}{4}$.

For each card, a guess can be either right or wrong. Using *R* for "probability of right" and *W* for "probability of wrong" and putting the first card guessed in bold font, we can see the possibilities as:

$$(\boldsymbol{R} + \boldsymbol{W})(R + W) = \boldsymbol{R}(R + W) + \boldsymbol{W}(R + W) = \boldsymbol{R}R + \boldsymbol{R}W + \boldsymbol{W}R + \boldsymbol{W}W$$

Because we are only interested in the total number of right or wrong guesses, the order of the cards does not matter. So, there is 1 way to guess both cards right, 2 ways to guess one card right and the other card wrong, and 1 way to guess both cards wrong. This result can be written as $(R + W)^2 = R^2 + 2RW + W^2$. Replacing *R* with its probability, $\frac{1}{2}$, and *W* with its probability, $\frac{1}{2}$, we get $R^2 = \frac{1}{4}$, $2RW = 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{2}{4}$, and $W^2 = \frac{1}{4}$. (Note that these fractions are not reduced on purpose).

1. For 3 cards, we would have $(R + W)^3 = R^3 + 3R^2W + 3RW^2 + W^3$. So, we can see that the probability of guessing all of the colors correctly is $R^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$. Replace the values for *R* and *W* in the right-hand side of the equation and explain the meaning of each term in the expression.

2. For four cards, write out the expansion of $(R + W)^4$. Calculate the probability of guessing all 4 cards right (or wrong).

3. Using your answer to question 2, what is the probability of correctly guessing the colors of *exactly 2* of the 4 cards?

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4. Complete the table below using the results from problems 1 – 3. In the third column, write the probabilities as a sum (do not reduce any fractions). Describe all of the patterns that you notice in the table.

Expression	Expansion	Probabilities
$(R + W)$	$R + W$	$\frac{1}{2}+\frac{1}{2}$
$(R + W)^2$	$R^2 + 2RW + W^2$	$\frac{1}{4}+\frac{2}{4}+\frac{1}{4}$
$(R + W)^3$		
$(R + W)^4$		

Look back at the cases for two cards. There are 2 ways to guess one card right and one card wrong because the correct card could be guessed either first or second. In the case of 3 cards, the possible outcomes are given by $(R + W)(R + W)(R + W)$. This result shows we can get two Rs and one W by picking the Rs from any 2 of the 3 factors and the W from the remaining factor.

The number of different ways to choose 2 objects from a group of 3 distinct objects, when order does not matter, is called the combination of 3 objects taken 2 at a time, and can be written as ${}_3C_2$. This is easy to calculate by hand for small numbers. For example, ${}_4C_2 = \dfrac{4!}{2!(4-2)!} = \dfrac{24}{(2)(2)} = 6$. So, the probability of guessing exactly 2 cards correct out of 4 would be ${}_4C_2 \cdot R^2W^2 = 6 \cdot \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{6}{16} = 0.375$.

Note that you can also use your calculator to find the number of combinations. For example, to find ${}_4C_2$, enter the command **4 nCr 2**. (To find **nCr** press [MATH] [◄] [3].)

5. Complete the following table:

Number of Cards	Probability of all correct	Probability of *exactly half* correct	Probability of all wrong
4	$\frac{1}{16}=0.0625$	$\frac{6}{16}=0.375$	$\frac{1}{16}=0.0625$
6			
10			
20			
30			

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6. In the long run, sheer guessing would lead to getting approximately half right and half wrong. So, for a large number of trials, one might expect the probability of guessing exactly half right and half wrong to be 50%. Based on your previous answer, does this seem true? Why?

For questions 7–9, suppose that a student takes a 25-question multiple-choice test, with four choices for each question.

7. Find the probability of guessing all 25 questions correctly. (Hint: In this case, $R = \frac{1}{4}$ and $W = \frac{3}{4}$.)

8. What is the probability of guessing all 25 questions incorrectly?

9. What is the probability of guessing *exactly 20* of the 25 questions correctly?

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Suppose that instead of figuring out the probability of a certain score *exactly*, one wishes to know the probability of *at least* a certain score. For example, in the case of the 25 question multiple-choice test, suppose one needs to answer 15 questions correctly in order to pass. What is the probability of guessing the answers and getting a passing score? (Hint: consider the expanded expression and which cases would have to be considered).

The coefficients of the binomial theorem can also be generated from Pascal's Triangle:

```
                1

            1       1
              \   /
        1       2       1
          \   /   \   /
    1       3       3       1
      \   /   \   /   \   /
1       4       6       4       1
```

Each row starts and ends with a 1, and the other numbers are the sum of the two numbers above. For example, the 6 in the final row shown is the sum of 3 + 3 from the preceding row. Note that the top row is called row 0. This means that counting starts with 0, which is consistent with $(R + W)^0 = 1$. Similarly, the first entry in row n is ${}_nC_0$.

Show that for row 4, the numbers in Pascal's Triangle are in fact equivalent to the number of combinations of 4 objects taken 0, 1, 2, 3, and 4 at a time.

Verify that ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$ for positive values of n and r. Explain how this relates to Pascal's Triangle.

Additional Resources

- This Web site gives additional information about the binomial theorem, including references to the theorem by Sherlock Holmes, Gilbert and Sullivan, and Monty Python: **http://en.wikipedia.org/wiki/Binomial_theorem**
- This Web site has an interactive game to test your ESP on a set of 5 cards. **http://www.mathagonyaunt.co.uk/STATISTICS/ESP/esp.html**

  The game keeps track of your score and tells you at each step the probability of doing as well as your cumulative score. In the case on the site, $R = 0.2$ and $W = 0.8$. Students can try their luck and compare their calculations with those of the computer.

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