NUMB3RS Activity: An Irrational Approach to Music
Episode: "The Running Man"

Topic: Simple Continued Fractions **Grade Level:** 9 - 12
Objective: Use simple continued fractions to approximate irrational numbers
Time: 20 - 30 minutes
Materials: graphing calculator

Introduction

In "The Running Man," Charlie talks about a willow flute that he built. A willow flute is typically a flute without finger holes and sounds are produced based on the natural harmonic scale when it is blown at different strengths. In his discussion, Charlie also mentions a pentatonic flute, which is another of the folk flute family. He discusses how a simple continued fraction can be used to approximate the irrational number needed to ensure that the flute will have an equal temperament.

An equal temperament means that the ratio of the frequencies of any two of the five adjacent notes is constant. Unlike the willow flute, the pentatonic flute has finger holes, and interestingly enough, deciding where to put the holes in this flute requires approximation – the ratio of consecutive frequencies is usually an irrational number! It is hard to believe that making beautiful music relies on compromise and approximation.

On a pentatonic flute the intervals of notes are composed of the minor third, the fourth, the fifth, the minor seventh, and the octave from a 12-note octave. With only 5 finger holes, this flute is simple in form but provides a sad and moving scale, known for its meditative qualities.

To build a pentatonic flute, one must approximate an irrational number to determine the placement of the holes. This type of approximation can be found using a simple continued fraction.

A simple continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

where a_0 is any integer, and a_1, a_2, a_3, … are positive integers. (To determine how to compute the set of constants for a number, see the extensions following the student activity.) When approximating *rational* numbers, this set of constants is finite, so the continued fraction is exact. However, with *irrational* numbers, the set of constants is infinite.

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In continued fractions, as the number of constants used increases, each successive fraction provides more accuracy. For example, you can approximate the irrational number $\sqrt{3}$ using a simple continued fraction with constants $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 1$. The first continued fraction used contains only the constant $a_0 = 1$ and approximates $\sqrt{3}$ as 1.

The next approximation, which uses the first two constants ($a_0 = 1$ and $a_1 = 1$) is $1+\frac{1}{1}=2$. Further approximations can be found by using more fractions and including more constants. Thus, the third and fourth approximations are $1+\cfrac{1}{1+\cfrac{1}{2}}=1.\overline{6}$ and $1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1}}}=1.75.$ As the number of constants involved in the simple continued fraction increases, the approximations get closer to the actual value of $\sqrt{3}$ (which is about 1.732).

Discuss with Students

1. The value of $\sqrt{2}$ is about 1.4142, and can be approximated with continued fractions using constants $a_0 = 1$, $a_1 = 2$, $a_2 = 2$, $a_3 = 2$. Using only the constant $a_0 = 1$, what is the first approximation for $\sqrt{2}$?

2. **a.** The second approximation for $\sqrt{2}$ uses the constants a_0 and a_1. What is the second approximation of $\sqrt{2}$? Write it using the continued fraction and in decimal form.

b. Which approximation (from Question 1 or Question 2a) is closer to the actual value of $\sqrt{2}$?

3. **a.** Find the third and fourth approximations of $\sqrt{2}$. Write them in both fractional and decimal form.

b. Which of the four approximations (from Questions 1, 2a, and 3a) is the closest to the actual value of $\sqrt{2}$?

4. How could you create an even better approximation for $\sqrt{2}$?

Note: The previous observations can be greatly generalized. Joseph Lagrange in 1770 proved that every quadratic irrational (square root) number has a periodic or repeating continued fraction. For example, the list of constants for $\sqrt{13}$ is:
[3, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, ...].

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Discuss with Students Answers:

1. *1* **2a.** $1+\frac{1}{2}=1.5$ **2b.** *the second approximation*

3a. *the third approximation is* $1+\cfrac{1}{2+\cfrac{1}{2}}=1.4$ *and the fourth approximation is* $1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}=1.41\overline{6}$.

3b. *the fourth approximation* **4.** *In general, the more constants used to calculate a continued fraction, the better the approximation of the number.*

Student Page Answers:

1. 3, $3+\frac{1}{7}=\frac{22}{7}\approx 3.1428$, $3+\cfrac{1}{7+\cfrac{1}{15}}=\frac{333}{106}\approx 3.1415$, and $3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1}}}=\frac{355}{113}\approx 3.14159$

2. 1, $1+\frac{1}{1}=2$, $1+\cfrac{1}{1+\cfrac{1}{1}}=1.5$, and $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}=1.\overline{6}$

3. *The numerators and denominators are successive Fibonacci numbers. This pattern continues*:
$\frac{13}{8}=1.625$, $\frac{21}{13}\approx 1.6154$, $\frac{34}{21}\approx 1.6190$, $\frac{55}{34}\approx 1.6176$ **4.** $2^{1/12}\approx 1.05946$ **5.** $\frac{89}{84}\approx 1.0595$

6.

Note	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
Approximate Frequency Ratios	1	1.059	1.123	1.189	1.260	1.335	1.415	1.499	1.588	1.682	1.782	1.888	2
Common Fraction Approximations	1		$\frac{9}{8}$		$\frac{5}{4}$	$\frac{4}{3}$		$\frac{3}{2}$		$\frac{5}{3}$		$\frac{15}{8}$	2

7a. *Because an even temperament is used, the frequency ratios are be equal for each pair of adjacent notes, regardless of which note you start on. So, the frequency ratios for each interval should not be changed.* **7b.** *1, 1.189, 1.335, 1.499, 1.782*

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Name: ______________________________ Date: _______________

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In "The Running Man," Charlie discusses a pentatonic flute. He indicates that a simple continued fraction is used to approximate the irrational number needed to ensure the flute has an equal temperament. An equal temperament means that the ratio of the frequencies of any two adjacent notes is constant. Finding where to place the holes when constructing the flute requires an approximation because the ratio of the frequencies of consecutive notes on the flute is an irrational number; that is, a non-repeating and non-terminating decimal.

A simple continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

where a_0 is any integer, and a_1, a_2, a_3, … are positive integers. To use a continued fraction to approximate the value of an irrational number, the first approximation requires only the constant a_0. The second approximation uses the constants a_0 and a_1. Each successive approximation uses an additional constant, making this an iterative process. In general, the more constants used to calculate a value of a continued fraction, the better the approximation of the number.

Simple Continued Fractions and Irrational Number Approximations

1. Find the first four approximations for the irrational number $\pi \approx 3.14159$, using a simple continued fraction with constants $a_0 = 3$, $a_1 = 7$, $a_2 = 15$, $a_3 = 1$.

2. The golden ratio, $\frac{1+\sqrt{5}}{2} \approx 1.618$, is an irrational number that can be approximated using a simple continued fraction with constants of $a_0 =1$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$. Find four approximations for the golden ratio. Write the results in both fraction and decimal form. (To learn more about the Golden Ratio and its application to art and nature, see the extensions page.)

3. Using $a_4 = 1$, the fifth continued fraction approximation for the golden ratio is $\frac{8}{5}$. Study the fractions that you calculated in Question 2, and describe any patterns that you see in the sequence of fractions. Use any found patterns to predict the next two fractions that you think would be in this sequence. Then test your prediction by actually computing the next two fractions using the constants $a_5 = 1$ and $a_6 = 1$.

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Simple Continued Fractions and Music

For each octave on a piano there are twelve notes. When a key on the piano is pressed, the key strikes a string that vibrates at a specific frequency. Because of the way the piano is tuned, the frequencies of different notes are related to each other. For example, the A below middle C has a frequency of 220 cycles per second (Hz) and the A above middle C vibrates 440 Hz (exactly double).

4. A piano is tuned using **equal temperament.** This means that the ratio of the frequencies of any two consecutive notes is always equal. What is this ratio? (Hint: We know that A below middle C has a frequency of 220 Hz, and there are 12 notes to the A above middle C, which has a frequency of 440 Hz. So, $220 \times r^{12} = 440$. The ratio is r.)

5. You can use continued fractions to approximate the value of r. Use a simple continued fraction with constants $a_0 = 1$, $a_1 = 16$, $a_2 = 1$, $a_3 = 4$ to approximate the value of r. Write your answer in decimal form rounding your answer to the nearest ten thousandth.

6. Use the value of r to complete the table below for one octave. Instead of using known frequencies, we will use ratios of frequencies based on the first note of the scale. (Note that because we are starting with C and ending with C, the ratio for the octave is 2:1.) In the first row, use a decimal approximation rounded to the nearest thousandth. In the second row, find a common fraction that could be used to approximate the decimal value. This should only be completed for the blank rows in the table. For example, the ratio of the frequency of D to the frequency of C is about 1.122, which could be approximated by $\frac{9}{8}$ (which is equivalent to 1.125).

Note	**C**	**C#**	**D**	**D#**	**E**	**F**	**F#**	**G**	**G#**	**A**	**A#**	**B**	**C**
Approximate Frequency Ratios	1	1.059	1.122										2
Common Fraction Approximations	1		$\frac{9}{8}$										2

Recall that a pentatonic flute does not have 12 notes in an octave. This flute has an octave consisting of 5 notes: E, G, A, B, and D. In musical terms, they are the intervals that are composed of the minor third, the fourth, the fifth, and the minor seventh from the twelve-note scale, *beginning with the note E*.

7a. A Musical Challenge The table in Problem 6 shows a scale starting with C. The table can be used for an octave starting with E for the pentatonic flute by placing the starting note (E) in the first column and copying the following notes in order.

Explain why the frequency ratios should not be changed.

b. The notes that can be played on a pentatonic flute are E, G, A, B, and D. What are the values of the ratios for the pentatonic flute?

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The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

For the Student

- To learn more about continued fractions and to learn how to write any number as a continued fraction, visit these Web sites.

  http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/cfINTRO.html#genCF
  http://en.wikipedia.org/wiki/Continued_fraction

  Using the information from these Web sites, find the constants needed to write a simple continued fraction for an irrational number such as $\sqrt{5}$, and for a rational number such as $\frac{7}{8}$. Describe the similarities and differences between the two continued fractions.

- This Web site provides 9 lessons where you can learn about ratios, including the "Golden Ratio" – a ratio of length to width that can be found in art, architecture, and nature.

  http://illuminations.nctm.org/LessonDetail.aspx?ID=L510

Additional Resources

- Visit this Web site for information about how to build your own pentatonic flute.

  http://www.flutespirit.com/PrivateLesson/Dvorak/FluteScale.html

- Read this article to learn more about different octaves and continued fractions.

  http://www.research.att.com/~njas/sequences/DUNNE/TEMPERAMENT.HTML

- Read the following article to learn more about other uses of continued fractions.

  Masunaga, David. "To Be Continued …" *Student Math Notes.* Reston, VA: National Council of Teachers of Mathematics, September 1993. Reprinted with more activities in *Teaching with Student Math Notes, Volume 3*, edited by Carol Findell. Reston, VA: The Council, 2000, pp. 111–118. The book can be purchased from their Web site at **http://www.nctm.org**.

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