EXPLORATIONS

CHAPTER 4

Solving Equations

We assume the reader has worked carefully through Chapters 1 and 2. In this chapter we will use various features of the TI-86 to investigate the solutions to a variety of equations, many of which we would have little chance to solve without the aid of the calculator.

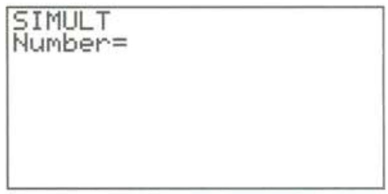
§1 – Solving a System of Linear Equations

Consider the following system of linear equations:

$$\begin{cases} x_1 - 2x_2 + x_3 = 4 \\ x_2 - 2x_3 = 1 \\ x_1 + x_2 + x_3 = 0 \end{cases}$$

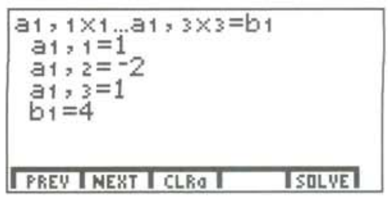
We will use the built-in simultaneous solver to find the solution to this system.

1. From the home screen, press [2nd] [SIMULT] to get (4.1.1).



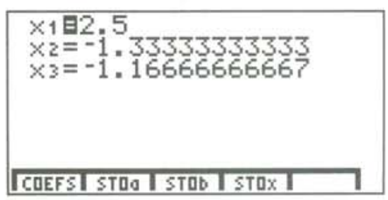
(4.1.1)

2. Note that we are first prompted for the number of equations (which must match the number of variables). Type [3] and press [ENTER]. Now enter the first equation by pressing [1] [ENTER] [(-)] [2] [ENTER] [1] [ENTER] [4]. See (4.1.2).



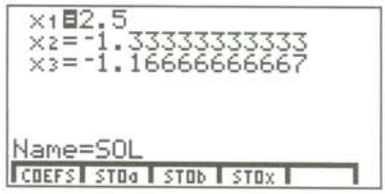
(4.1.2)

3. Press [ENTER] to record these coefficients and move on to the other two equations. We can use the ⟨**PREV**⟩ and ⟨**NEXT**⟩ menu items to recheck any entries that have been made. Now select ⟨**SOLVE**⟩ and obtain (4.1.3).



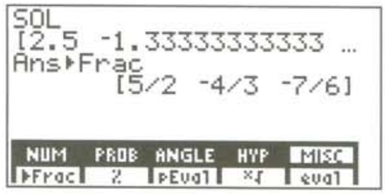
(4.1.3)

4. The ⟨**STOx**⟩ menu item enables us to store this solution as a vector. We will store this solution in the vector named *SOL*. Figure (4.1.4) results from (4.1.3) by selecting ⟨**STOx**⟩ and typing in *SOL*.



(4.1.4)

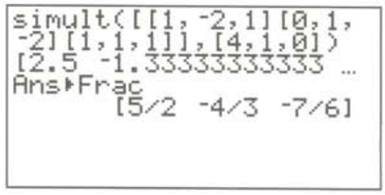
5. Pressing [ENTER] will complete the assignment of the solution vector to the variable name *SOL* and will produce (4.1.3) again. Now exit to the home screen and check, as in (4.1.5), that *SOL* is, in fact, the desired vector. We have also used the ▸**FRAC** feature in the MATH MISC menu to convert our solution to fraction form.



(4.1.5)

This problem also can be approached from the home screen using matrices.

First, access **simult** through the CATALOG/VARIABLES menu. Complete the command as shown at the top of (4.1.6), and press [ENTER]. Pressing [2nd] [ENTRY] [2nd] [ENTRY] will recall the **Ans ▸FRAC** command which can then be used to convert the solution to fraction form.

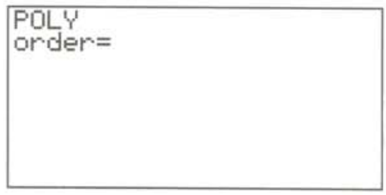


(4.1.6)

§2 – Finding Roots of Polynomials

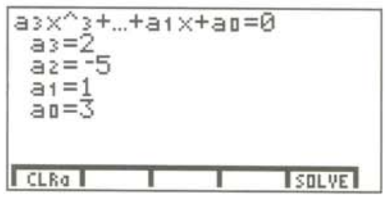
The built-in POLY function can be used to solve polynomial equations of degree two or higher, up to degree thirty. Consider $2x^3 - 5x^2 + x + 3 = 0$.

1. Press [2nd] [POLY] to obtain (4.2.1).



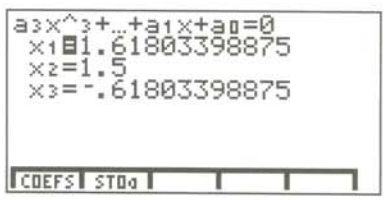
(4.2.1)

2. Enter 3 as the order and enter the coefficients of the polynomial expression as in (4.2.2).



(4.2.2)

3. Select ⟨**SOLVE**⟩ to compute the three real solutions to this equation as in (4.2.3). The two menu items of (4.2.3) indicate that we can look back at the coefficients of the equation or store the coefficients as a list.



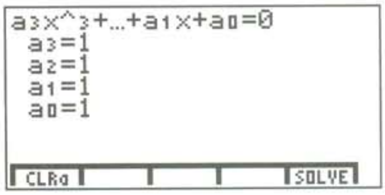
(4.2.3)

We can also store the values of the roots using the [STO▸] key. Let's store x_2 as *R*2. Use the down arrow to place the cursor on x_2. Press [STO▸] and enter *R*2. Then press [ENTER]. If you like, exit to the home screen and check to see that *R*2 has value 1.5.

§3 – Finding Complex Roots

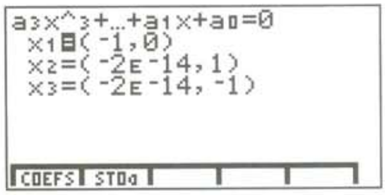
The TI-86 can do complex arithmetic, so it will find complex as well as real roots for polynomial equations. Consider the equation $x^3 + x^2 + x + 1 = 0$.

1. Solving this equation using [2nd] [POLY], as in (4.3.1), will produce (4.3.2).



(4.3.1)

2. Noting that the TI-86 form of the complex number $a + bi$ is (a, b), we see that the solutions for the given equation are -1 and $-2\text{E}-14 \pm i$. In fact, the solutions are almost certainly exactly -1 and $\pm i$. Note, however, if any of the roots are complex, then the TI-86 reports all the roots in the form of complex numbers.

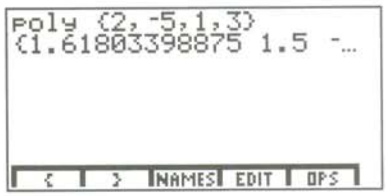


(4.3.2)

§4 – Finding Polynomial Roots from the Home Screen

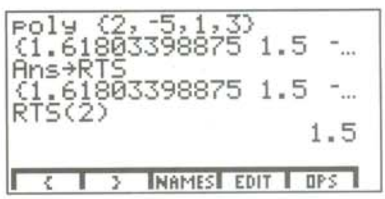
The TI-86 also gives the option of solving polynomial equations from the home screen.

1. In (4.4.1) we use the **poly** function found in the CATLG submenu of the CATALOG/VARIABLES menu and a list of the coefficients to again solve the polynomial equation $2x^3 - 5x^2 + x + 3 = 0$. *(Press* [2nd] [LIST] *to get the menu at the bottom of the screen.)* Note that the result will be a list containing the same three roots we found in §2. See (4.2.3) for comparison.



(4.4.1)

2. In (4.4.2) we have stored the list of roots in the variable *RTS*. Also note that we have entered *RTS*(2) to access the second entry of this list. The other two roots can be accessed in a similar manner.

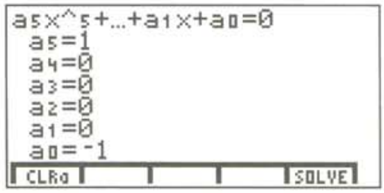


(4.4.2)

§5 – Finding nth Roots of Unity

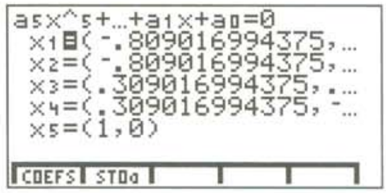
One problem often studied in elementary trigonometry is that of finding n^{th} roots of unity. The equation solving capabilities of the TI-86 make this process rather easy.

1. For example, the fifth roots of unity are the solutions to $x^5 - 1 = 0$. In (4.5.1) we have entered the appropriate coefficients in the **POLY** editor.



(4.5.1)

2. In (4.5.2) we have selected ⟨**SOLVE**⟩ to find these five roots.



(4.5.2)

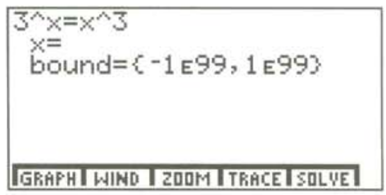
It is a worthwhile exercise to compare these solutions with the complex numbers given by

$$\cos\left(\frac{2\pi k}{5}\right) + i\sin\left(\frac{2\pi k}{5}\right), \textit{ for } k = 0,\ 1,\ 2,\ 3,\ 4.$$

§6 – Solving Equations Using the Built-In SOLVER Function

To solve equations which are not polynomial in nature we will make use of the built-in SOLVER function. The ROOT and ISECT functions of Chapter 2 both make use of SOLVER.

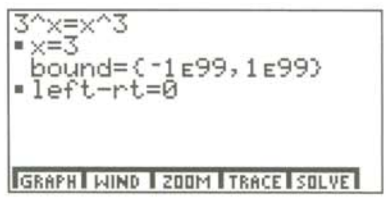
1. Consider the equation $3^x = x^3$. Press [2nd] [SOLVER] and enter the equation $3^x = x^3$. Press [ENTER] to obtain a display similar to (4.6.1).



(4.6.1)

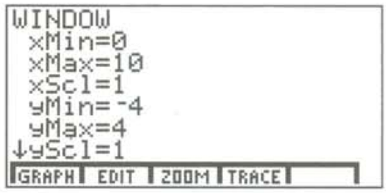
The variable x has whatever value it had in the most recent computation involving x. The expression "bound = {–1E99, 1E99}" indicates that the calculator will attempt to find a solution to the equation with $-1\text{E}99 < x < 1\text{E}99$. This interval bound can be edited but any initial value for x must satisfy the bound conditions before selecting the menu item ⟨**SOLVE**⟩ since SOLVER uses the displayed x-value as an initial guess for the solution.

2. If no value is given for x then the midpoint of the bound interval will be used as an initial guess. Let's see what happens if we give no value for x and just select ⟨**SOLVE**⟩. The result is given in (4.6.2) and, of course, 3 is a solution for this equation.



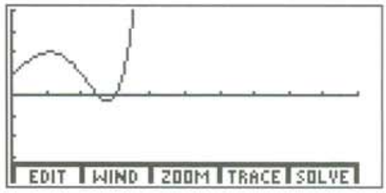
(4.6.2)

3. We can also graph from the SOLVER editor. First select ⟨**WIND**⟩ and set the window values as in (4.6.3).



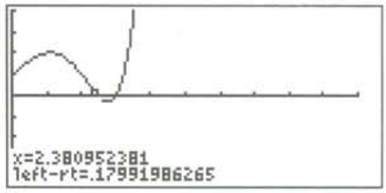
(4.6.3)

4. Then select ⟨**GRAPH**⟩ to obtain (4.6.4). This is a graph of $y = 3^x - x^3$. That is, SOLVER graphs the left side of the original equation minus the right side. Since the graph in (4.6.4) shows two x-intercepts, the original equation has two solutions for $0 \le x \le 10$.



(4.6.4)

5. To find a value for the smaller of these two solutions, select ⟨**TRACE**⟩ and then press the left arrow enough until the trace cursor is as in (4.6.5). Press EXIT ⟨**EDIT**⟩ and notice that the trace value of x will now be the initial guess.

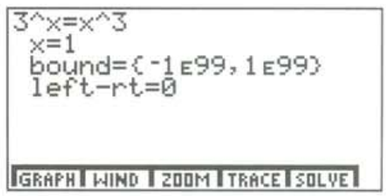


(4.6.5)

6. Select ⟨**SOLVE**⟩ to get (4.6.6) and see that $x = 2.4780526802884$ is a second solution to $3^x = x^3$.

(4.6.6)

7. The SOLVER algorithm is definitely sensitive to the initial guess for x. For example, if we take $x = 1$ for an initial guess as in (4.6.7)



(4.6.7)

and select ⟨**SOLVE**⟩, the result is (4.6.8) indicating that the algorithm failed to locate a solution.

(4.6.8)

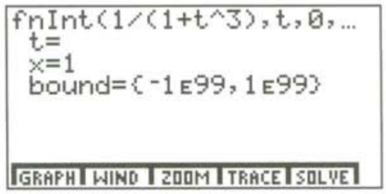
§7 – Solving an Integral Equation

Now consider the equation

$$\int_0^x \frac{1}{1+t^3}\,dt = 0.8.$$

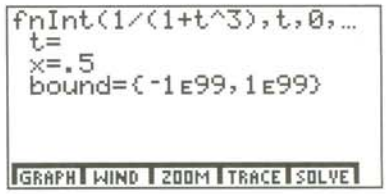
Note: *If you are not familiar with the* **fnInt** *command and how the* **tol** *setting controls its accuracy, it is recommended that you look at §5 and §8 of Chapter 6 before proceeding with this section.*

1. After doing so, with **tol** having the default value of 0.00001, if we enter the above equation in SOLVER using the **fnInt** command and press [ENTER], we get a screen similar to (4.7.1). The full entry would be **fnInt**(1/(1+t^3),t,0,x) = 0.8.



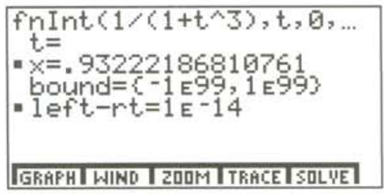
(4.7.1)

2. Notice that the variable t is listed along with x, but t is just the dummy variable of integration. In this case, we can leave t blank and enter an initial guess for x, say $x = 0.5$ as in (4.7.2).



(4.7.2)

3. Select ⟨**SOLVE**⟩. After a bit of a wait, we see that the calculator reports a solution of $x = 0.93222186810761$ in (4.7.3).

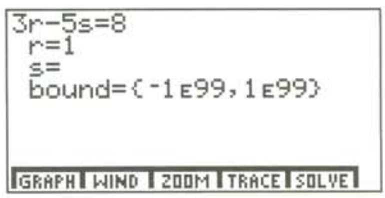


(4.7.3)

§8 – Multivariable Equations

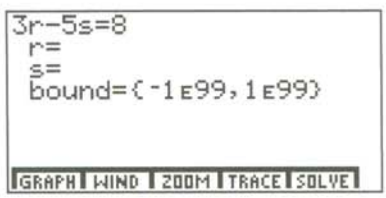
In §7, there is the indication that SOLVER can deal with equations involving more than one variable. We will explore this a little with the simple equation $3r - 5s = 8$.

1. Enter this equation in the SOLVER editor and press [ENTER] to obtain a display similar to (4.8.1).



(4.8.1)

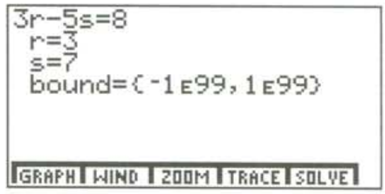
2. If values are shown for r and s, they are just the values left over from previous computations involving these variables. Clear both values as in (4.8.2).



(4.8.2)

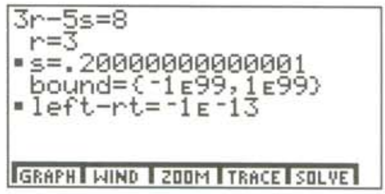
Solving Equations *(Continued)*

3. Now enter 3 for r and 7 for s. See (4.8.3).



(4.8.3)

4. If we select ⟨**SOLVE**⟩ while the cursor is on s, then the calculator treats r as a constant having value 3 and solves $3r - 5s = 8$ for s, correctly obtaining $s = 0.2$ as in (4.8.4).

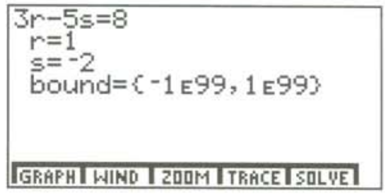


(4.8.4)

If we recreate (4.8.3) and select ⟨**SOLVE**⟩ while the cursor is on r, then the calculator treats s as a constant with value 7, and solves for r to obtain $r = 14.333$. In §9 we will experiment with SOLVER's ⟨**GRAPH**⟩ menu item in connection with the equation $3r - 5s = 8$.

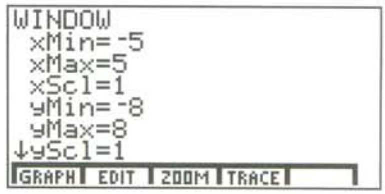
§9 – Using the SOLVER Graph Menu

1. Set values for r and s as in (4.9.1).



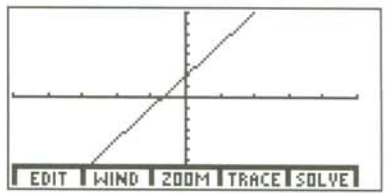
(4.9.1)

2. Select ⟨**WIND**⟩ and set the window variables as in (4.9.2).



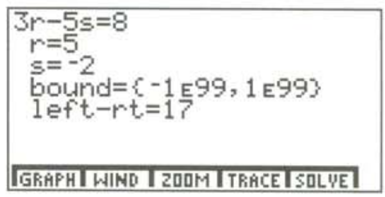
(4.9.2)

3. Select ⟨**EDIT**⟩ to return to (4.9.1). Place the cursor on the $r = 1$ line, and select ⟨**GRAPH**⟩. The resulting graph in (4.9.3) will be the graph of $3r - 5s - 8$ (left minus right), with r treated as the variable and s treated as a constant having value –2. So (4.9.3) should be the graph of $y = 3r + 10 - 8$, or $y = 3r + 2$.



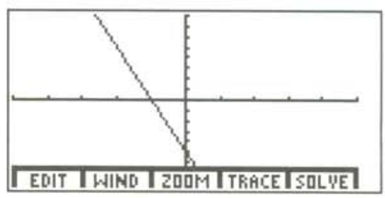
(4.9.3)

4. Select ⟨**EDIT**⟩ to obtain (4.9.4). Notice that the current value of r is the same as **xMax**, indicating again that r was treated as the independent variable for graphing.



(4.9.4)

5. Change r to have value 1, and place the cursor on the $s = -2$ line. Select ⟨**GRAPH**⟩. This time the calculator treats r as a constant with value 1 and graphs the line $y = 3 - 5s - 8$, or $y = -5s - 5$, with s now playing the role of the independent variable. See (4.9.5).



(4.9.5)

§10 – Using SOLVER to Find Zeros of a Function

In this section we will see that SOLVER can find zeros of functions without formally being given an equation. Since this section involves an equation with a trig function, make certain that the calculator is in the radian mode.

1. Using [CLEAR], clear the equation line of the SOLVER editor so that we start with (4.10.1).

(4.10.1)

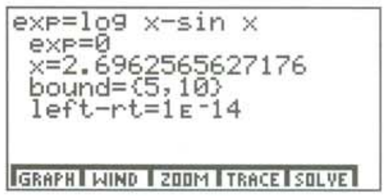
2. Enter the expression $\log x - \sin x$ and press [ENTER] to obtain a display similar to (4.10.2). Notice that SOLVER has in a sense created its own equation, namely $exp = \log x - \sin x$, by creating the variable *exp*. So now we treat the problem as an equation with two variables x and *exp*.

(4.10.2)

3. To find a zero of the function given by $f(x) = \log x - \sin x$, we assign the variable *exp* the value zero, place the cursor on the x line, and set x equal to 3. Then select ⟨**SOLVE**⟩ to obtain (4.10.3).

(4.10.3)

4. Since the function $\log x - \sin x$ has more than one zero, this is a case in which modification of the bound set may be appropriate. For example, set bound = {5, 10} as in (4.10.4).

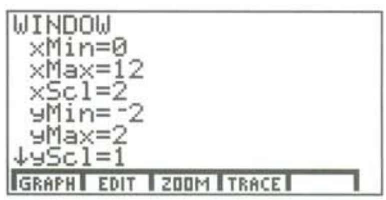


(4.10.4)

5. Place the cursor on the x = 2.6962565627176 line and select ⟨**SOLVE**⟩. The result is (4.10.5) because the initial guess for x must be within the bound set. Select ⟨**GOTO**⟩, change x to have value 6 and select ⟨**SOLVE**⟩. We find that x = 7.3283477786913 is a solution to $\log x = \sin x$ which lies between 5 and 10.

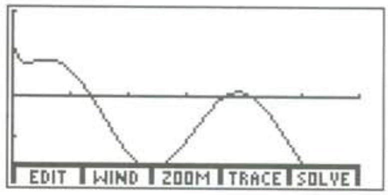
(4.10.5)

6. Now set the WINDOW as in (4.10.6) and select ⟨**GRAPH**⟩.



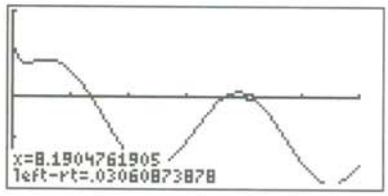
(4.10.6)

7. This produces the graph of $y = \sin x - \log x$ in (4.10.7). It appears that $\log x = \sin x$ has three solutions.



(4.10.7)

8. Select ⟨**TRACE**⟩ and move the cursor along the curve until it is near the largest of the three x-intercepts as in (4.10.8).



(4.10.8)

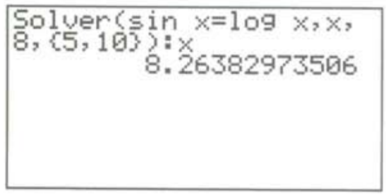
9. Press [EXIT], and select ⟨**SOLVE**⟩. The x trace value becomes the initial guess for x, and eventually, we get (4.10.9), with the third solution to $\log x = \sin x$ being 8.263829735064.

(4.10.9)

§11 – Using SOLVER from the Home Screen

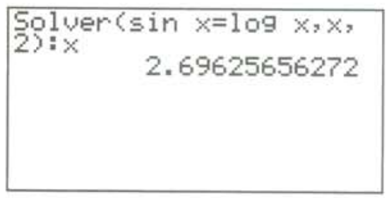
Similar to POLY, SOLVER also can be used from the home screen.

1. After exiting to the home screen, in (4.11.1) we ask the calculator to find a solution to $\sin x = \log x$ in the interval [5, 10], starting with initial guess $x = 8$. Actually, the colon in (4.11.1) indicates that we are asking the TI-86 to do two things, namely use SOLVER and then report what value it found for x.



(4.11.1)

2. Figure (4.11.2) shows that the bounding set for x is optional for SOLVER.



(4.11.2)

§12 – Solving a Finance Problem

In §12 and §13 we will consider some finance problems. We will go to the home screen to enter and store an equation to be recalled into SOLVER.

1. Clear the home screen and enter

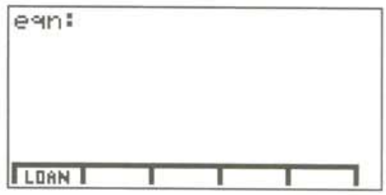
$$LOAN = AMT = \left(\frac{1-(1+rate/12)^{-months}}{rate/12}\right) \times PMT$$

(4.12.1)

as in (4.12.1), remembering to press [ENTER] when finished. In this expression, *AMT* is the amount of the loan, *rate* is the annual percentage rate, *PMT* is the monthly payment, *months* is the number of months over which the loan will be repaid, and *LOAN* is just a variable name for the equation

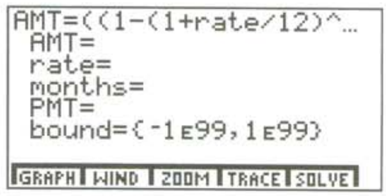
$$AMT = \left(\frac{1-(1+rate/12)^{-months}}{rate/12}\right) \times PMT.$$

2. Now press [2nd] [SOLVER], and clear the equation line of the SOLVER editor. Notice that menu item F1 contains *LOAN*. See (4.12.2).



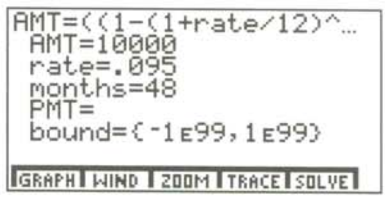
(4.12.2)

3. Now press [2nd] [RCL] ⟨**LOAN**⟩ [ENTER] [ENTER] to obtain (4.12.3). It is important to note here that we are using [2nd] [RCL] to obtain (4.12.3). Selecting ⟨**LOAN**⟩ after clearing the SOLVER equation line will not give the same screen as is shown in (4.12.3).



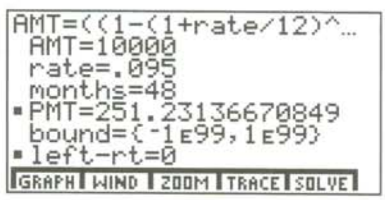
(4.12.3)

4. We are now dealing with an equation with four variables. If we give values for three of the variables SOLVER will find a value for the fourth variable. It is optional to give an initial guess for the fourth variable. For example, (4.12.4) shows the setup needed to find the monthly payment required to pay off a $10,000 loan in 48 months with an annual percentage rate of 9.5%.



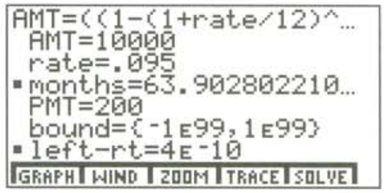
(4.12.4)

5. Place the cursor on the *PMT* line and select ⟨**SOLVE**⟩. The result is given in (4.12.5).



(4.12.5)

6. In similar manner, (4.12.6) shows that the person who can pay only \$200 per month will need 64 months to pay off this \$10,000 loan.

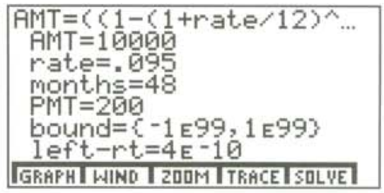


(4.12.6)

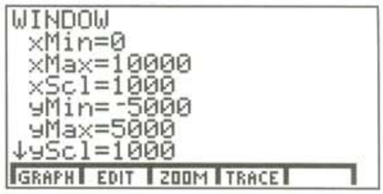
§13 – Another Use of the SOLVER Graph Menu

We will now use the ⟨**GRAPH**⟩ menu item in the SOLVER editor in connection with the *LOAN* equation introduced in §12.

1. With the screen as in (4.13.1) and the cursor on the *AMT* line, select ⟨**WIND**⟩ and set the WINDOW values as in (4.13.2).



(4.13.1)

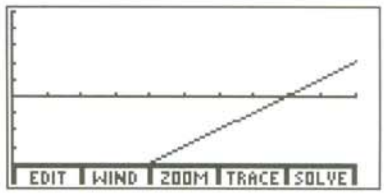


(4.13.2)

2. Then select ⟨**GRAPH**⟩ to obtain the graph of

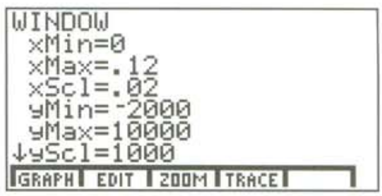
$$y = AMT - \left(\frac{1 - (1 + rate/12)^{-months}}{rate/12} \right) \times PMT$$

where *rate*, *months*, and *PMT* are treated as constants and *AMT* is the variable. So the graph will be a line. See (4.13.3).



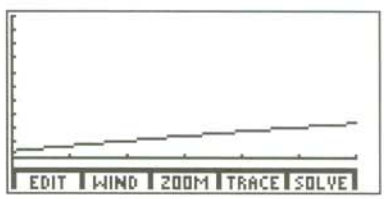
(4.13.3)

3. The horizontal intercept indicates that at 9.5% annual percentage rate, 48 monthly payments of \$200 each will pay off a loan of approximately \$8,000. Return now to the screen (4.13.1) by selecting ⟨**EDIT**⟩, and place the cursor on the *rate* line. Set the WINDOW as in (4.13.4).



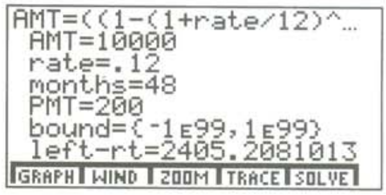
(4.13.4)

4. Select ⟨**GRAPH**⟩ to get (4.13.5). The graph is again that of the above equation, but now the variable will be *rate* and all other quantities are constant. The fact that this graph has no horizontal intercept means that 48 monthly payments of \$200 each will not pay off the \$10,000 loan no matter how small the interest rate is. Of course, this is a predictable result since $200 \times 48 = 9600$.



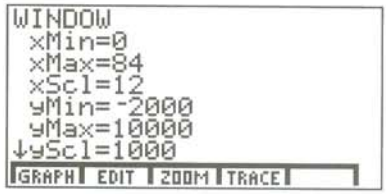
(4.13.5)

5. Finally, from (4.13.5) select ⟨**EDIT**⟩ to obtain (4.13.6).



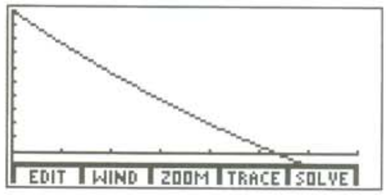
(4.13.6)

6. Reset *rate* to have value 0.095, and put the cursor on the *months* line. Select ⟨**WIND**⟩, and set the WINDOW values as in (4.13.7).



(4.13.7)

7. Figure (4.13.8) shows the resulting graph of the equation with *months* treated as the variable and all other quantities being fixed with the values given in (4.13.7). The horizontal intercept indicates that there is a number of months for which monthly payments of \$200 will pay off the \$10,000 loan if the annual percentage rate is 9.5%.



(4.13.8)

§14 – Solving a Matrix Equation

We close this chapter by using the SOLVER feature to study the nature of the solution set to the matrix equation given by

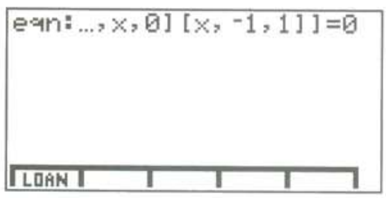
$$\det\begin{bmatrix} 1 & 1 & x \\ -1 & x & 0 \\ x & -1 & 1 \end{bmatrix} = 0.$$

It is certainly possible here to symbolically compute the determinant above and algebraically find all the solutions to this equation, but we will use the graphing feature of the SOLVER to study the nature of the solutions first. *(The interested reader may want to take a brief detour to the beginning of Chapter 7 to see how to enter a matrix on the TI-86 and where to find the determinant function.)*

1. Enter the above equation in the SOLVER by entering the equation as

```
det [[1,1,x][-1,x,0][x,-1,1]] = 0
```

to obtain (4.14.1).



(4.14.1)

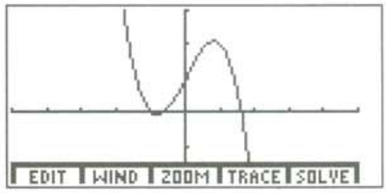
2. Pressing ENTER will give (4.14.2).

(4.14.2)

3. Set the WINDOW variables as indicated in (4.14.3).

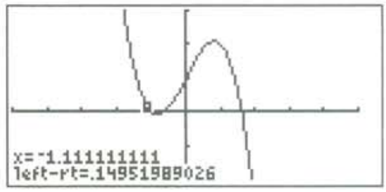
(4.14.3)

4. Select ⟨**GRAPH**⟩ to obtain (4.14.4). It is apparent that there are three solutions to the matrix equation.



(4.14.4)

5. Select ⟨**TRACE**⟩ and move the cursor to a position just to the left of the most negative solution as in (4.14.5).



(4.14.5)

6. Press EXIT and select ⟨**EDIT**⟩ to obtain (4.14.6). Notice that the trace value of x now appears as the current value of x.

(4.14.6)

7. Selecting ⟨**SOLVE**⟩ will produce (4.14.7) and lead to the conjecture that $x = -1$ is a solution to the equation.

(4.14.7)

In a similar manner, we can discover that the other solutions are $x = 1.61803398875$ and $x = -\,0.61803398875$. *It is a worthwhile exercise to show, algebraically, the exact solutions to the equation given above are* $x = -1$ *and* $x = \left(1 \pm \sqrt{5}\right)/2$.

Exercises

1. Solve the following system of equations

$$\begin{cases} 2x_1 + x_2 - x_3 = 1 \\ \quad x_2 + x_3 = 4 \\ -x_1 + 2x_2 \quad = 5 \end{cases}$$

2. Find the fifth roots of 32 using POLY. Compare the results with the answer to Exercise 3 of Chapter 3.

3. Consider the equation $xy = 6e^{(x+y-5)}$.

   (a) Find all solutions to this equation for which $x = 2$.

   (b) Find all solutions to this equation for which $y = 1$.

   (c) Find all positive solutions to this equation for which $y = 2x$.

4. As with the finance problem considered in §12, find the monthly payment required to pay off a \$15,000 loan in seven years if the annual percentage rate is 8.4%.

5. A two-digit number can be represented in the form $10A + B$, where A is the ten's digit and B is the units digit. Use SOLVER to find the only two-digit number which is equal to the square of the sum of its two digits.

6. Use SOLVER to find the first three positive solutions to $e^{-x} = \sin x$.