

How Long



Answers

7 8 9 10 11 12



TI-84CE



Investigation



Student



60 min

Problem

What is the average distance between two points selected randomly within a square with side lengths of 1 unit? To solve this question using analytic methods would require mathematical techniques well beyond high school mathematics courses; however this problem can be solved approximately using simulations, the greater the number of simulations the greater the accuracy of the result. The purpose of this activity is to illustrate how solutions to such problems can be reduced into much simpler versions using simulations.

Note: Students will need access to the HOWLONG program available from the activity download.

Question: 1.

Based on the information above estimate the average length of a line drawn between two random points located inside a square with side lengths 1cm.

Answers will vary – students should realise that the average would most likely be greater than 0.5 since the length of a line could vary between 0 and $\sqrt{2}$.

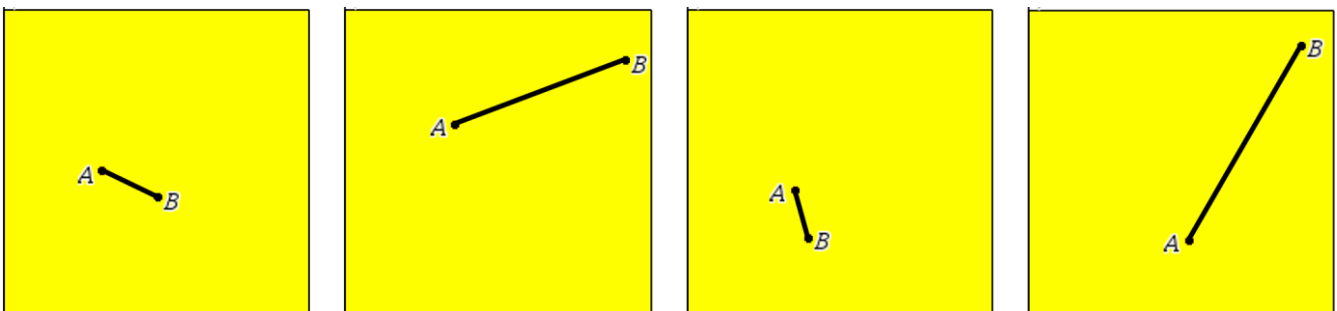
The purpose of this question is to raise some discussion over what seems like a relatively simple question. Another interesting thing to do at this point is to take an average of the class estimates 'wisdom of the crowd'.

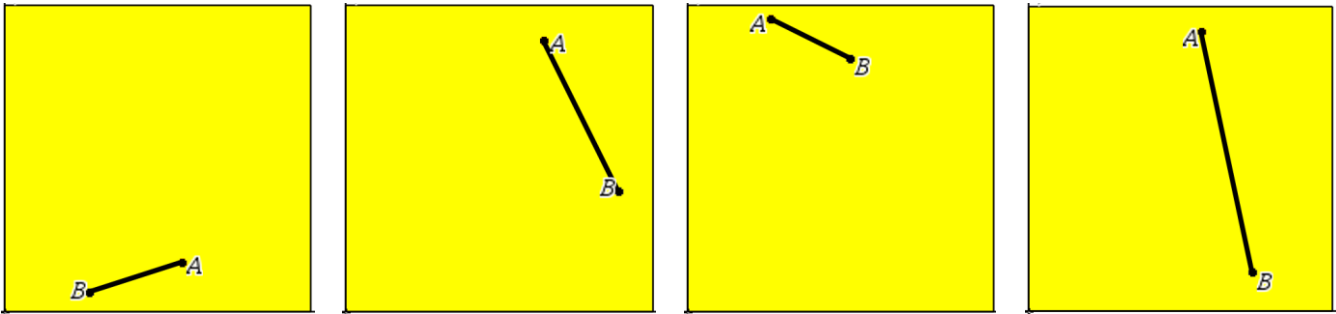
Question: 2.

To begin here are eight samples where two points A and B have been randomly generated so that they land inside the square. The square has been scaled up to 4cm \times 4cm to make it easier to measure the length of line segment AB. Measure the length of the line AB and divide it by 4 to scale it back down, record the result for each square and then compute the average length.

Answers will vary but should be between 0.45 and 0.49cm.

Students that write answers close to 15 have forgotten to divide by the scale factor: 4.



**Question: 3.**

The minimum length of a line segment produced using this method would be 0cm. What would be the longest possible length?

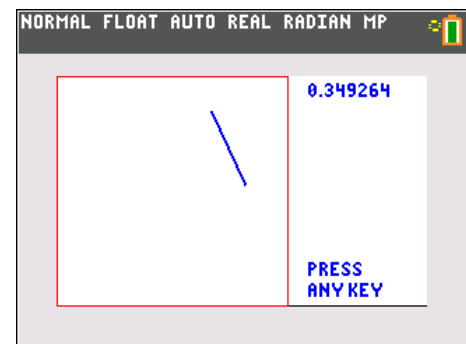
Longest length occurs when A and B are in opposite corners, using Pythagoras's theorem: $\sqrt{2}$ cm

Using a Simulator

Run the program: **HowLong**

You will be prompted to select the number of lines you want to generate.

The program pauses between after each line is drawn and automatically computes the length.

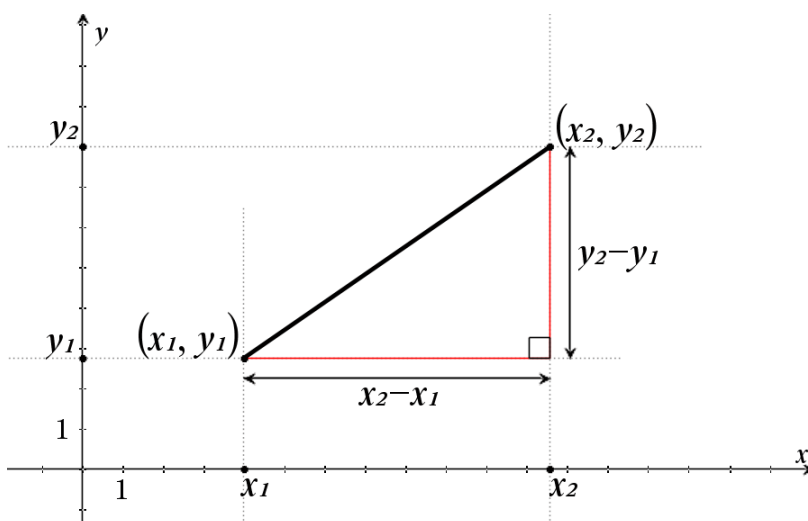
**Question: 4.**

Record 20 lengths and calculate the average of these lengths.

Answers will vary: Mean length will most often be between: 0.45 and 0.59cm.

Creating Samples

The simulation on the calculator uses the Cartesian plane as a platform. Two points are plotted making it easy to produce random points. The rand() command produces a random number between 0 and 1, so two points can be generated by plotting $x_1 = \text{rand}()$, $y_1 = \text{rand}()$ and $x_2 = \text{rand}()$, $y_2 = \text{rand}()$. The length of the line can then be calculated by computing the distance between the two points.

**Question: 5.**

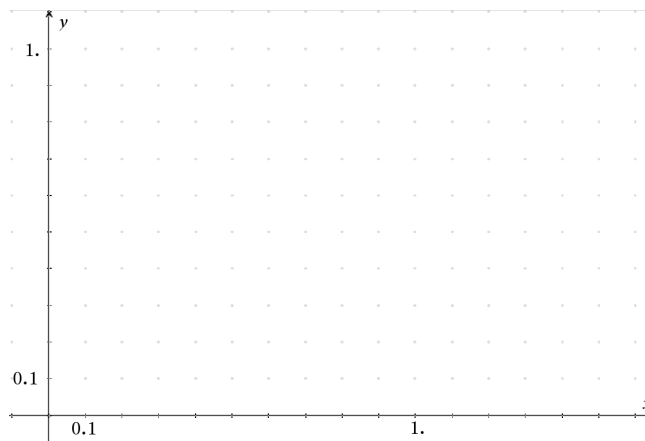
Use the diagram shown opposite and Pythagoras's theorem to derive a formula for the distance between two points.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Question: 6.

Use the calculator to generate 4 random numbers corresponding to x_1, y_1, x_2, y_2 . Record the points and plot them on the Cartesian plane. Use your formula to calculate the distance between the two points.

Answers will vary



Creating a LARGE sample

The random number generator can be used to generate large quantities of random numbers. Type the following command to generate 100 random numbers and store them in L_1 .

$$\text{rand}(100) \rightarrow L_1$$

Repeat this process to store 100 random numbers in lists L_2, L_3 and L_4 .

NOTE: The numbers shown opposite represent a sample only; your random numbers should be different.

In the Stat-List editor navigate to the top of List 5.

Assuming L_1 represents x_1 ; $L_2 = y_1$, $L_3 = x_2$ and $L_4 = y_2$, type your distance formula at the top of list 5.

The start of the formula is shown opposite.

NOTE: If you place your formula in “ ” (quotation marks) it will automatically update when values in the dependents lists are changed. This will however slow the calculator down a little as the calculator recomputes all 100 distances each time a list is changed.

Once the distances have been calculated in L_5 , return to the home screen and use the LIST menu to calculate the mean of the values (lengths) stored in L_5 .

This result is the mean of 100 random lines placed in a 1cm x 1cm box.

```
NORMAL FLOAT AUTO REAL RADIAN MP
rand(100)→L1
{0.7586270698 0.714864000▶
rand(100)→L2
{0.3046465107 0.042397550▶
rand(100)→L3
{0.3222455898 0.881601974▶
rand(100)→L4
{0.4576946135 0.394414158▶
```

L1	L2	L3	L4	L5	S
0.7586	0.3046	0.3222	0.4577	-----	
0.7149	0.0424	0.8816	0.3944		
0.3063	0.7114	0.6546	0.4012		
0.3326	0.7534	0.8776	0.3287		
0.8185	0.1555	0.6679	0.2832		
0.5942	0.416	0.322	0.4408		
0.5045	0.0774	0.0722	0.4783		
0.7596	0.2302	0.9528	0.0662		
0.6031	0.6983	0.3424	0.6869		
0.6304	0.063	0.6048	0.6542		

$L_5 = \sqrt{(L_1)}$

```
NORMAL FLOAT AUTO REAL RADIAN MP
NAMES OPS MATH
1:min(
2:max(
3:mean(
4:median(
5:sum(
6:prod(
7:stdDev(
8:variance(
```

Question: 7.

Record the results of 10 sample means. Use this data to estimate the mean length of a line randomly placed inside a 1cm × 1cm square.

Answers will vary but will mostly vary between 0.45 and 0.59.

Question: 8.

Suppose the side lengths of the square are doubled in length, how would this affect the average length? Adjust all relevant formulas and record the mean for 10 samples. Use this data to determine the mean length of a line randomly generated inside a 2cm x 2cm square.

HINT: $2 \times \text{rand}()$ will produced random numbers between 0 and 2.

As the original square has been dilated by the same amount in all directions so too will the average length of the line, therefore the average should be double that for a 1cm × 1cm square.

Question: 9.

Conduct an investigation into the average length of a line placed inside a rectangle with dimensions 1cm x 2cm.

Students may change the random number formula in columns x_1 and x_1 OR y_1 and y_2 but not both. The average length is approximately: 0.804772.

The formula for computing the average distance between two points randomly placed in a

rectangle is given by: $\frac{1}{15} \left(\frac{L_w^3}{L_h^2} + \frac{L_h^3}{L_w^2} + d \left(3 - \frac{L_w^2}{L_h^2} - \frac{L_h^2}{L_w^2} \right) + \frac{5}{2} \left(\frac{L_h^2}{L_w} \log \left(\frac{L_w + d}{L_h} \right) + \frac{L_w^2}{L_h} \log \left(\frac{L_h + d}{L_w} \right) \right) \right)$

where $d = \sqrt{L_w^2 + L_h^2}$.

Extension Investigation

Suppose two random points are placed inside a box measuring 1cm × 1cm × 1cm. What would be the average length of the line connecting these two points?

The average distance between two points in a 1cm × 1cm × 1cm cube is approximately: 0.66170 also referred to as Robbins Constant.¹

¹ David P. Robbins – Robbin's is also known for his work on cyclic pentagons where each side length is a rational number (The total area is also rational) One of the unsolved problems of mathematics is to prove that the diagonals of Robbin's pentagons are always rational.