RATE OF CHANGE

by C. C. Edwards
Coastal Carolina University, Conway, SC
Edwards@coastal.edu
(Please feel free to email me questions and /or comments.)

Key Topic: Derivatives

Abstract:

This activity is designed for students who have already learned the definition of the derivative and that it can be used determine the slope of a curve. The goal of the activity is to show these students that the derivative can also be used to determine the rate of change of one variable with respect to another. This is achieved by investigating the relationship between average and instantaneous velocity. Students will also learn the physical meaning of negative and positive rates of change.

Prerequisite Skills:

- Knowledge of the limit definition of the derivative.
- Ability to evaluate the derivative of a polynomial.

Degree of Difficulty: Easy

Needed Materials: TI-89

NCTM Principles and Standards:

- Content Standards – Algebra
    - Represent and analyze mathematical situations and structures using algebraic symbols
    - Use mathematical models to represent and understand quantitative relationships
    - Draw a reasonable conclusion about situation being modeled

- Process Standards
    - Representation
    - Connections
    - Problem Solving

RATE OF CHANGE

At some time in your educational experience, you most likely had it drummed into your head that "*distance equals rate times time*." But were you really taught the truth?

Consider this scenario: You are driving on an interstate which has a 60 mph speed limit. After 2 hours of driving, you find that you have traveled 116 miles. Did you ever exceed the speed limit during this trip?

According to the "*distance equals rate times time*" formula, you traveled at the nice, safe speed of 58 mph, and thus never exceeded the speed limit. But is this plausible? If you set cruise control to 58 mph and never experienced any traffic, then yes, it is plausible. But a more realistic scenario is that at least once during that 2 hour drive, you came upon a slow moving truck which you had to pass. You may even have exceeded the speed limit while doing this! So you most likely were not always traveling at 58 mph, and you may have indeed exceeded the speed limit.

So what does this say about the "*distance equals rate times time*" formula you learned? It says that one very important key word was left out of this formula – the word *average*. This formula gives you only the *average* rate (*average* speed, *average* velocity). It's like test grades. If on four math tests you get a 28, 100, 100, and 100, and a friend of yours gets an 84, 84, 80, and 80, you both have an average of 82. But who's the better student?

So why was the word *average* left out when you learned this formula? It's because you need calculus to understand the concept of an *instantaneous* rate. It's like the speedometer in a car. If you are driving down an interstate trying to stay at a speed limit of 60 mph and you don't have cruise control turned on, you will notice that the speedometer probably gives you readings which vary between 58 and 62 mph. How does a speedometer do this? It uses very small time units, measured in seconds as apposed to hours, to calculate the *average* speed. *Instantaneous* rate takes this a step further. As you will see, *instantaneous* rate is found by allowing the time interval to approach zero!

To illustrate this, let's get back to our original scenario: You are driving on an interstate which has a 60 mph speed limit. After 2 hours of driving, you find that you have traveled 116 miles. Did you ever exceed the speed limit during this trip?

Well, to answer this question, we need to know a bit more about what you were doing during these 2 hours. So let's assume that the function $f(x) = 4x^4 - 48x^3 + 109x^2$ gives the number of miles you traveled by time x, measured in hours. This is graphed at the right for $0 \le x \le 2$ and $0 \le y \le 120$.
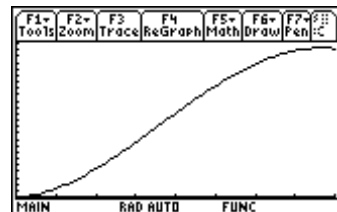
To find out if you were speeding, we could use tables (see p. 225 of the ***TI-89 Guidebook***) to help us calculate *average* speed by entering time intervals of 15 minute (.25 of an hour), and then having the TI-89 find the *average* speed. To fine tune this process, we could change time to 10 seconds, and then 5 seconds, and then maybe even 1 second. But we are not here to teach you how to do tables! So enter $4x^4 - 48x^3 + 109x^2$ into **y1**, and let's find a direct calculation of the *average* speed. The formula for *average* speed is

$$average \text{ speed} = \frac{y1(t_2) - y1(t_2)}{t_2 - t_1}$$

where time is measured in hours and *average* speed is measured in mph.

If, for example, you wanted to find the *average* speed during the first hour, you would store 0 in the variable t1 and store 1 in the variable t2. And then you would enter the formula for *average* speed as shown at the right. (**y1** can be found in the **Var-Link** menu which is accessed by pressing 2nd =.)
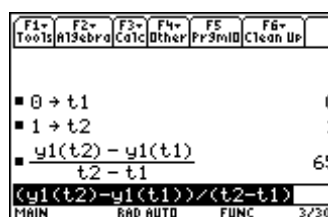
After pressing enter, you find that the *average* speed was 65 mph. But this indicates that you were exceeding the 60 mph speed limit during the first hour of travel! When were you speeding?

To answer this question, let's first test the first half hour of travel by storing 0.5 in t2. (Since time is measured in hours in the **y1** formula, t2 must be entered as a fractional part of an hour.) Instead of re-entering the formula for *average* speed, simply press the up arrow key until the formula for *average* speed is highlighted, as pictured at the right. Pressing ENTER will place the formula on the command line, and pressing ENTER again will evaluate it using the new value for t2, thus finding that the *average* speed during the first half hour was 43 mph. So it is unlikely, but not impossible, that you were speeding during the first half hour of travel.
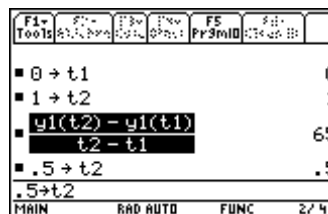
QUESTION 1. In the following, as above, $y1(x) = 4x^4 - 48x^3 + 109x^2$ denotes the number of miles you traveled by time *x*, measured in hours.
  a. What was the *average* speed during the second half hour of travel?
  b. What was the *average* speed during the time frame from ½ to ¾ of an hour?
  c. What was the *average* speed during the time frame from ¾ to 1 hour?
  d. What was the *average* speed during the third half-hour of travel?
  e. What was the *average* speed during the last 15 minutes of travel?

Based on your answer to part a, it appears that you were speeding sometime during the second half-hour of travel. Your answers to parts b and c indicate that you were most likely speeding during the whole second half hour of travel. And your answer to part d says that at least at some (maybe many) point in time you were speeding. But what about the last 15 minutes of travel? An *average* speed of slightly less than 60 mph does not, on the surface, exonerate you from speeding! Maybe you hit a 6 minute traffic jam which caused you to travel at an average of 30 mph, and then you made up for it by speeding at 80 mph for the remaining 9 minutes!

So how do we determine how fast you were going at any point in time? We could look at the *average* speed over a time period of 1 hr, 30 mins, 15 mins, 10 mins, 1 min, 30 seconds, 1 second, etc. Or we could use the concept of a limit and look at a time frame which goes to zero. In terms of the notation given in the above example, this would give us the following formula:

$$\text{``speed''} = \lim_{(t2-t1)\to 0} \frac{y1(t2) - y1(t1)}{t2 - t1}$$

In general mathematical terms, if $f(x) =$ the distance traveled by time x, then the "speed" for a time period from $t1 = x$ to $t2 = x + h$, where $h > 0$, is

$$\text{``speed''} = \lim_{((x+h)-x)\to 0} \frac{f(x+h) - f(x)}{(x+h) - x}.$$

Simplified, this gives

$$\text{``speed''} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

"Speed" is in quotation marks because it is not *average* speed. It is what is now called *instantaneous* speed. And from its definition, you should recognize that *instantaneous* speed at time $x = a$ is simply $f'(a)$.

So in the above scenario, how fast were you going when you were 1 hour and 50 minutes into your trip? Since time is measured in hours, the answer is $f'(1\frac{5}{6})$ where

$f(x) = 4x^4 - 48x^3 + 109x^2$, the distance traveled in x hours.

How can we get the TI-89 to find the *instantaneous* speed of $f'(1\tfrac{5}{6})$? Well that is really easy since we already have $f(x)$ defined in **y1**. Go to the home screen by pressing [HOME]. Press [F3] **1** to find a derivative. Enter **y1** (**y1** can be found in the **Var-Link** menu which is accessed by pressing [2nd] [=].) Since **y1** is a function of x, press [X] and close the parentheses. Press [,] and then [X] to indicate that the derivative is to be taken with respect to the variable x. Then press [)] to indicate that you are finished giving the calculator the command to find the derivative. To get the TI-89 to evaluate this derivative at $x = 1\tfrac{5}{6}$, press the evaluation key [|] (left column, fourth from bottom) and enter $x = 1 + 5/6$. To convert the fractional answer to a decimal, press [♦][ENTER]. The answer tells us that you were probably caught in traffic at that time, traveling at a paltry speed of slightly over 14 mph.
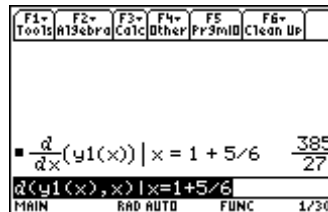
Note: To find the derivative when the function is not entered using [Y=], just enter the function in place of **y1** in the directions specified above. For example, the illustration at the right shows you how to find $f'(3)$ where $f(x) = x^2$.

SUMMARY:

- *average* rate $= \dfrac{\text{distance traveled}}{\text{time required to travel that distance}}$

- *instantaneous* rate (at time t) $= f'(t)$

  where $f(x) = $ distance traveled by time x.

CAUTION: Henceforth, in *all* calculus and calculus based physics textbooks, the word "rate" will mean *instantaneous* rate. If the *average* rate is desired, the word "rate" will be preceded by the adjective "*average*". So the word "rate" alone refers to the *instantaneous* rate, which is a derivative. This is a very important point since it appears to contradict the "*distance equals rate times time*" formula which you are used to.

ANSWERS:

1. a. 87 mph.
   d. 77 mph

   b. 83.3125 mph
   e. approx. 60 mph

   c. 90.68.75 mph