NUMB3RS Activity: Trawling for an Intersection
Episode: "Spree, Part II – Daughters"

Topic: Parametric Equations **Grade Level:** 11 - 12
Objective: Deriving and solving a system of parametric equations
Time: 15 - 20 minutes
Materials: TI-83 Plus/TI-84 Plus graphing calculator

Introduction

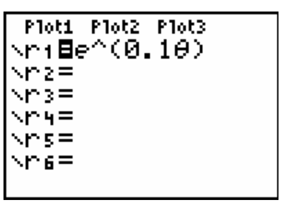
In "Spree, Part II," the FBI is chasing a criminal on the run. Don believes the criminal is within a certain bounded area in Los Angeles. Charlie compares the situation to a famous problem known as the Trawler Problem. This problem involves a fast boat chasing a slow boat until the slow one disappears into a fog bank. (In Charlie's case, the FBI is the fast boat, and the fog bank is the boundary of the dragnet.) The Trawler Problem assumes that the slow boat enters the fog bank, turns at a particular angle, and then continually heads in that direction.

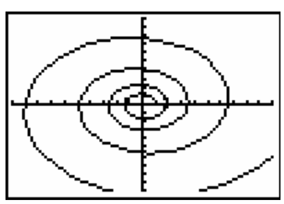
The surprising solution to the Trawler Problem is for the fast boat to proceed to the point where the boats would have met if the slow boat had made a 180° turn and headed back towards the fast boat. (Note that the slow boat would never actually do this, but it is a necessary assumption for the solution of the problem.) From that point, the fast boat should spiral out logarithmically. The boats will then intersect before the fast boat completes one full turn.

This activity is a continuation of the *NUMB3RS* activity "Spiraling Out." While "Spiraling Out" explored the many different types of spirals that can be graphed, this activity focuses on solving the Trawler Problem using parametric equations. The intent of this activity is to expose students to a question involving parametric equations that does not appear in most traditional textbooks.

Discuss with Students

Students should already know how to graph parametric equations as well as how to measure angles in radians. As a beginning example, have the students set their calculators to polar mode (press MODE and select **POL**) and graph the logarithmic spiral $r = e^{0.1\theta}$ using the window settings x: [–10, 10], *y*:[–10, 10], and θ:[0, 8π].





This spiral is similar to the one the fast boat will follow when pursuing the slow boat. The spiral is called logarithmic because if the equation $r = e^{0.1\theta}$ is solved for theta, the function is logarithmic.

A neat fact: draw any ray out from the center of the logarithmic spiral and let P_i be the points of intersection of that ray with the spiral. If tangent lines are drawn at each of these points, the angle between the ray and the tangent lines are always equal

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Student Page Answers:
1. *The default setting of the calculator is to graph polar equations as a function of* θ, *not as a function of r*.
2a. *(cos C, sin C)* **2b.** *(2 cos C, 2 sin C)* **2c.** *((t + 1) cos C, (t + 1) sin C)* **3a.** *a = 1* **3b.** *t + 1*
3c. $b = \frac{\ln(t+1)}{t}$ **4a.** *around 2.3 time units. (about 11.5 minutes)* **4b.** *around 3.66 time units (about 18.5 minutes).*

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Name ____ Date _______________

NUMB3RS Activity: Trawling for an Intersection

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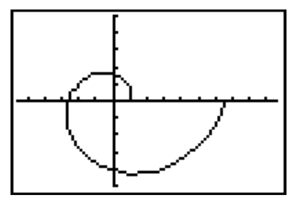
The surprising solution to the Trawler Problem is for the fast boat to proceed to the point where the boats would have met if the slow boat had made a 180° turn and headed back towards the fast boat. From that point, the fast boat should spiral out logarithmically. The boats will then intersect before the fast boat completes one full turn.

How can this be possible? The fast boat can catch the slow boat within one full turn without seeing where it is going? It is true and here's how.

The logarithmic spiral shown at the right is a polar equation that can be used to represent the path of the fast boat. If the path of the slow boat could also be graphed, it would be an easy matter of finding the point of intersection using the calculator. Unfortunately, some lines are impossible to graph in polar mode on the graphing calculator.



1. The polar equation that represents the path of the slow boat is an angle (i.e. $\theta = \frac{3\pi}{4}$.) Why is it impossible to graph that line in polar mode on your calculator? ____________________

Although you cannot use polar graphs to solve this problem, it is possible to graph the path of both boats at the same time using parametric equations. The units within the problem can be changed, but we will choose values that make the calculations easier. Assume that the slow boat is moving at a speed of 1 mile every 5 minutes and let *T* represent the number of 5-minute time units that have passed.

When the slow boat disappears into the fog bank at the origin, the fast boat is at point *N*. The fast boat then proceeds to where the small boat *would* be if it had turned around 180° and proceeded toward the fast boat. (Note that the slow boat would not actually do this, because it would be caught. But in order to solve the problem, this is a necessary assumption). For simplicity, assume that the fast boat takes 5 minutes to travel from point *N* to the assumed intersection point, *P*. When the chase into the fog bank begins, which we shall call time *T* = 0, the fast boat is at point *P*(1, 0).

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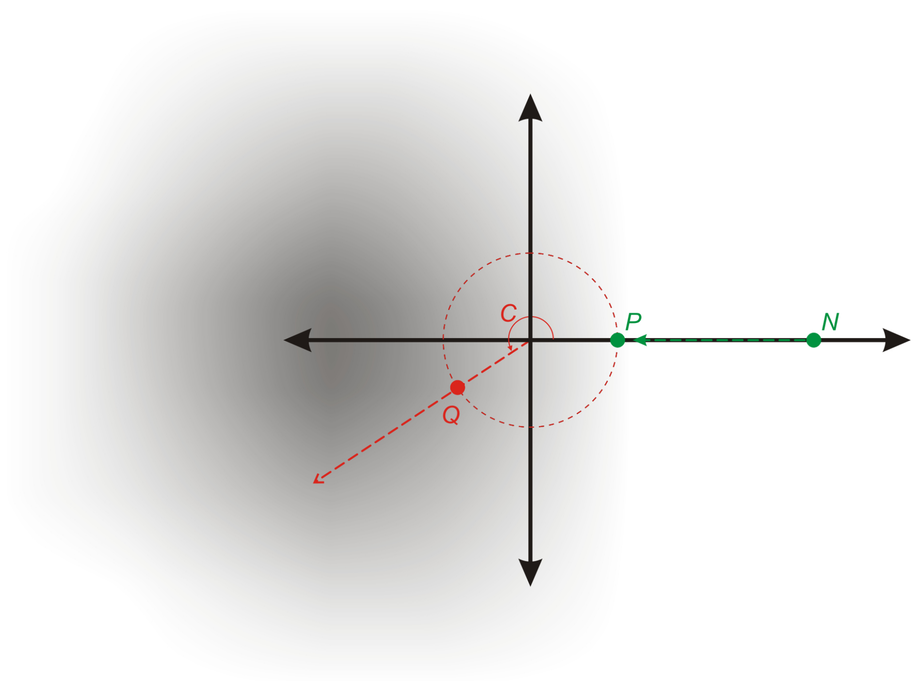
During the 5 minutes that the fast boat travels from *N* to *P*, the slow boat will travel 1 mile from the spot where it disappeared into the fog bank (the origin). The picture below shows the slow boat at point *Q* after 5 minutes, but depending on the angle *C* that the slow boat took, it could be anywhere on the circle of radius 1 mile.



2. **a.** Let *C* be the angle with the positive x-axis (measured in radians) at which the slow boat is traveling. What are the rectangular coordinates of the slow boat at time *T* = 0 in terms of *C*? ____________________

b. When *T* = 1, the slow boat is 2 miles from its starting point. What are the coordinates of its position in terms of *C*?

____

c. What are the coordinates of the slow boat at time *T* = *t*? _________________________

At time *t*, the **slow boat's** position can be described by the parametric equations below.

$$x = (t+1)\cos C$$
$$y = (t+1)\sin C$$

Using the equation for a logarithmic spiral $r = ae^{bt}$, the parametric equations for the fast boat at time *t* will be:

$$x = ae^{bt}\cos t$$
$$y = ae^{bt}\sin t$$

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To determine the values of *a* and *b* for this problem, the polar form of the equation of the path of the fast boat must be explored.

3. a. At time $t = 0$, the radius r must be 1 (because that is where the fast boat starts). Substitute $t = 0$ and $r = 1$ into the equation $r = ae^{bt}$ and determine the value of *a*.

____

b. In the equation $r = ae^{bt}$, substitute the value for *a* and rewrite the equation.

____

c. The fast boat is moving so that it is the same distance from the origin as the slow boat. At time $T = t$, how far from the origin is the fast boat?

____

d. Using the equation $r = e^{bt}$ with $T = t$ and $r = t + 1$, solve for *b*.

____

e. Substitute the values of *a* and *b* into the parametric equations. What are the resulting equations?

$x =$ _______________________

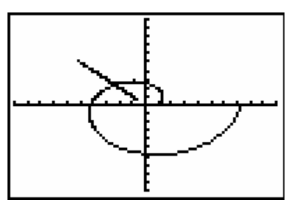
$y =$ _______________________

Using your graphing calculator, set the mode to parametric and use the following window settings: *x*:[–10, 10], *y*:[–10, 10], and *T*:[0, 2π]. Assuming that the slow boat takes a direction of $\frac{3\pi}{4}$, enter these parametric equations:

Slow boat: $X = t\cos\frac{3\pi}{4} + \cos\frac{3\pi}{4}$, $Y = t\sin\frac{3\pi}{4} + \sin\frac{3\pi}{4}$

Fast boat: $X = (t+1)\cos t$, $Y = (t+1)\sin t$

The resulting graph is shown at the right.



4. a. Using the TRACE key, determine the time *T* when the fast boat will intercept the slow boat. ____________________________________

b. Suppose the slow boat had taken an angle of $\frac{7\pi}{6}$. Determine the time *T* when the fast boat intercepts the slow boat. _____________________________________

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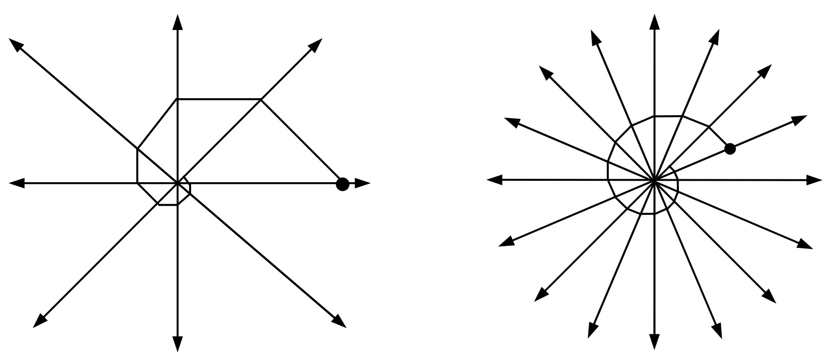
The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Activity: Drawing Logarithmic Spirals

For the Student

To construct a logarithmic spiral, draw six equally spaced rays emanating from a single point. Starting at a point along one ray, draw a perpendicular to a neighboring ray. Continue this process until all rays have been used. The perpendicular segments form a spiral. As the number of rays is increased to 10, 20, 30, etc., the sequence of segments approaches a smooth logarithmic spiral. The figures shown below show spirals drawn from 8 and 16 rays. (Mathematical Reflections: In a Room of Many Mirrors, Hilton et al. 1997, pp. 2-3).



Additional Resources

- Learn about a dog that is able solve complex pursuit problems in his head and obtain the quickest route to a ball thrown into Lake Michigan.
  http://www.sciencenews.org/articles/20040626/mathtrek.asp

- View artwork done using logarithmic spirals at the Web site
  http://www.uwgb.edu/dutchs/symmetry/log-spir.htm

- Read a mathematical paper on the pursuit problem for wheeled vehicles at the Web site
  http://lims.mech.northwestern.edu/~lynch/IGERT499/marshall-francis-tac2004.pdf

- Other spirals that can be drawn with the TI-83 Plus/TI-84 Plus graphing calculator are explored in the *NUMB3RS* activity "Spiraling Out." To download this activity, go to **http://education.ti.com/exchange** and search for "7449."

- Another way to draw logarithmic spirals is shown in the *NUMB3RS* activity "The Four Bug Problem: Step on No Pets." To download this activity, go to **http://education.ti.com/exchange** and search for "7423."

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