Investigating Properties of Kites

Definition:

Kite—a quadrilateral with two distinct pairs of consecutive equal sides (Figure 1).

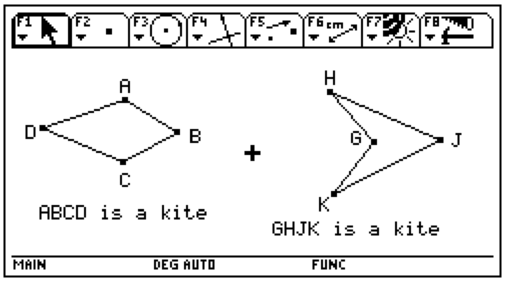

Figure 1

Construct and Investigate:

1. Determine three ways to construct a kite by using the Voyage™ 200. Test your constructions by dragging independent points of the kite to be sure that the figures remain kites under all conditions.
2. For each construction, determine the conditions under which the kite is convex and nonconvex.
3. Conjecture as to the type of quadrilateral formed by connecting consecutive midpoints of the four sides of a kite. Test your conjecture by constructing such a quadrilateral, using a kite formed by one of your methods. Does your conjecture hold true when the kite is nonconvex?
4. Draw the diagonals of a kite. List as many properties as possible that appear to be true about the diagonals of kites. Be sure to indicate properties that are true for both convex and nonconvex kites, as well as properties that are true for only convex kites.
5. Investigate the more specialized kites: the rhombus and the square. Verify that all properties of kites also appear to be true for these quadrilaterals.

Explore:

1. Construct a kite using one of your methods. Measure its area. Is there a relationship between the lengths of the diagonals of a kite and the area of the kite? Test your conjecture on a variety of kites by dragging vertices of the kites. Write an explanation that proves your conjecture is true for all kites. Does your method hold true when the kite is nonconvex? Is this relationship between the diagonals and area true for any other types of quadrilaterals?
2. Inscribe kite ***EFGH*** in rectangle ***ABCD*** so that ***H*** and ***F*** are at the midpoints of $\overline{AB}$ and $\overline{CD}$ respectively (Figure 2).

   What relationship exists between the area of the kite and the area of the rectangle? How does this relationship change as you slide ***E*** and ***G*** along the sides of the rectangle? If the area changes, when does it reach its maximum value? Explain.

   As you slide ***E*** and ***G*** along the sides of the rectangle, how does the perimeter of the kite change? If the perimeter changes, when does it reach its smallest value? Explain.

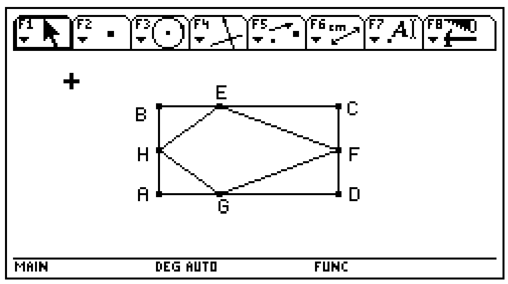

Figure 2

Teacher's Guide: Investigating Properties of Kites

Construct and Investigate:

1. This part of the exploration may be quite challenging for students, depending on their mathematical background, creativity, and ability to persevere during trial and error. Collaborative efforts among groups of students are appropriate for this activity. Finding more than one way to accomplish a task often requires students to think at higher cognitive levels. It also results in a better understanding of the concepts being studied.

   The following are among the methods that can be used to construct a kite:

   - Construct two intersecting circles. The quadrilateral that connects the centers of the circles to the points of intersection of the circles is a kite.
   - Construct a segment and its perpendicular bisector. Place two distinct points on the perpendicular bisector, and connect these points with the endpoints of the segment to form a kite.
   - Construct two isosceles triangles that share a common base and have two distinct third vertices. The pairs of equal sides of the isosceles triangles form a kite (excluding the base).
   - Construct two segments that share a common endpoint and do not form a right angle. Draw a reflection line through the two endpoints that are not shared between the segments. Reflect the two segments over this line. The resulting quadrilateral is a kite.
   - Construct a rectangle. Choose the midpoints of two opposite sides of the rectangle as two vertices of the kite. Draw a line parallel to the two chosen sides. This line will intersect either the rectangle or the extension of two sides of the rectangle. The quadrilateral formed by connecting these intersection points to the midpoints of the two sides of the rectangle is a kite.
   - Reflect any triangle over any one of its sides. The two remaining sides and their reflections form a kite.

2. Referring to the methods described in part 1 above, the kite will be nonconvex *if*

   - the centers of the circles are on the same side of the segment that connects their points of intersection.
   - the two points on the perpendicular bisector are on the same side of the segment.
   - the two isosceles triangles share a common base, but one lies inside the other.
   - the two noncommon endpoints are on the same side of the line that is perpendicular to the reflection line that contains the common endpoint of the segments.
   - the parallel line intersects the extensions of the sides of the rectangle and not the sides themselves.
   - the altitude to the reflection side crosses an extension of the side of the triangle and not the side itself.

3. The quadrilateral that connects consecutive midpoints of a kite will always be a rectangle (Figures 3 and 4).

   Draw either diagonal of a kite. The midsegments of the triangles formed by the diagonals of a kite are parallel to the base (the diagonal) and, therefore, are parallel to each other. This argument holds true for both diagonals, producing two pairs of congruent parallel lines. The result is an inscribed parallelogram.

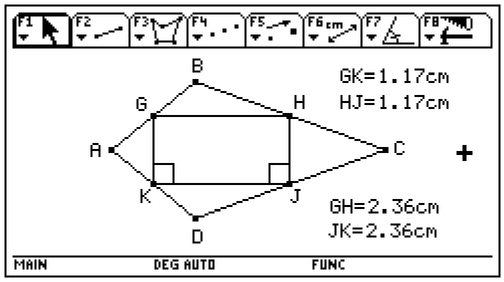

Figure 3

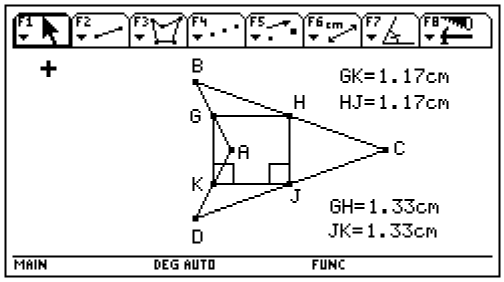

Figure 4

   One way to define a kite is the sides of two isosceles triangles sharing a common base. By this definition, the diagonal connecting the third vertex of each isosceles triangle is the perpendicular bisector of the base, the other diagonal. Therefore, the diagonals are perpendicular and the parallelogram formed by the midsegments is a rectangle (Figures 5 and 6).

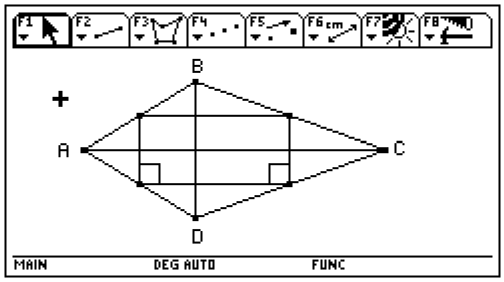

Figure 5

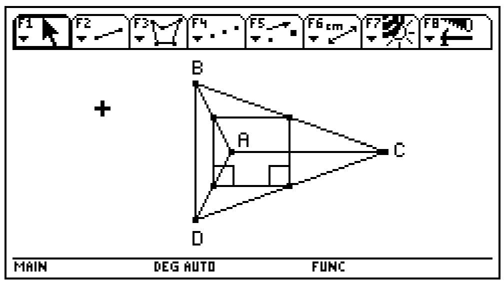

Figure 6

4. The diagonals of a kite are always perpendicular. In kite ***ABCD***, $\overline{AC}$ bisects $\triangle$***BAD***, $\triangle$***BCD***, and $\overline{BD}$, and lies on the line of symmetry of the kite. One-half the product of the lengths of the diagonals equals the area of the kite (Figure 7).

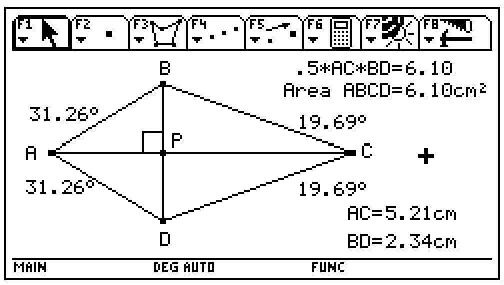

Figure 7

5. All properties of kites are also properties of rhombi and squares because these figures are special cases of kites. (Students may wish to find properties of these figures that are not properties of kites.)

Explore:

1. The area of a kite is always equal to one-half the product of the lengths of its diagonals (Figures 7 and 8). Because $\triangle ABC \cong \triangle ADC$, the two triangles have the same area. Together these two triangles make up the area of kite ***ABCD***.

   Because $\overline{AC}$ bisects $\overline{BD}$, $BP = \frac{1}{2} BD$.

   The area of $\triangle ABC = \frac{1}{2} AC * BP$ (Figure 7).

   Thus, the area of kite $ABCD$ = 2 * the area of $\triangle ABC = 2 * \frac{1}{2} * AC * \frac{1}{2} * BD = \frac{1}{2} AC * BD$.

   This is true for convex as well as nonconvex kites. Because both a rhombus and a square are kites, the areas of these figures equal one-half the product of their diagonals. You can prove this algebraically using symbolic algebra on the Voyage™ 200.

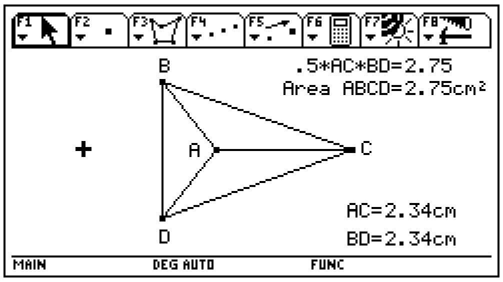

Figure 8

2. The area of the kite is always one-half the area of the rectangle, regardless of the location of the moving vertices of the kite (Figure 9). In this construction, $\overline{HF}$ is parallel and equal to $\overline{AD}$, the length of the rectangle; $\overline{EG}$ is parallel and equal to $\overline{AB}$, the width of the rectangle.

   By substitution, the area of kite $EFGH = \frac{1}{2} * AD * AB$. This is also exactly half the area of rectangle ***ABCD*** because the area of rectangle $ABCD = AD * AB$.

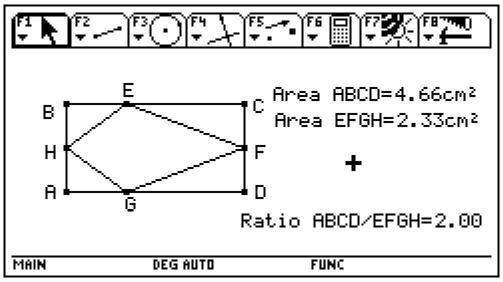

Figure 9

   The perimeter of kite ***EFGH*** will change as vertices ***E*** and ***G*** are moved. The perimeter minimizes when these vertices are moved to the midpoints of the two sides of the rectangle, forming a rhombus (Figure 10). The perimeter maximizes when ***G*** coincides with ***A*** or ***D*** as the kite becomes an isosceles triangle.

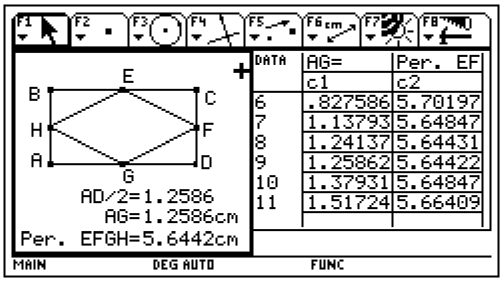

Figure 10

   Figure 11 shows a quadratic function (computed using the **QuadReg** tool) that models the data gathered on the length of $\overline{AG}$ and the perimeter of ***EFGH***.

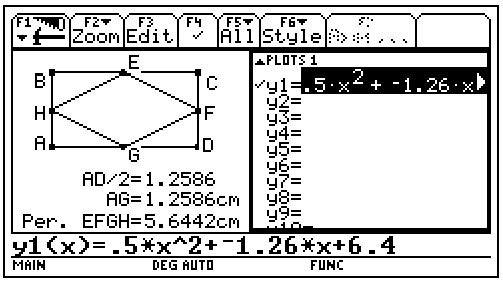

Figure 11

   Figure 12 shows the graph of the quadratic function in Figure 11, drawn over the scatter plot of the data. The minimum perimeter of kite ***EFGH*** is approximated by the minimum point of the function.

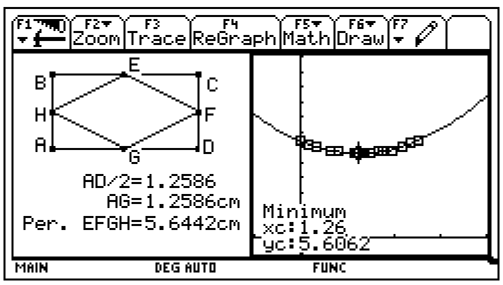

Figure 12

Investigating Properties of Trapezoids

Definitions:

Trapezoid—a quadrilateral with at least one pair of parallel sides (Figure 1).

Isosceles trapezoid—a trapezoid with one pair of base angles and the non-parallel pair of sides equal in measure.

Tessellation—a covering, or **tiling**, of a plane with a pattern of figures so there are no overlaps or gaps.

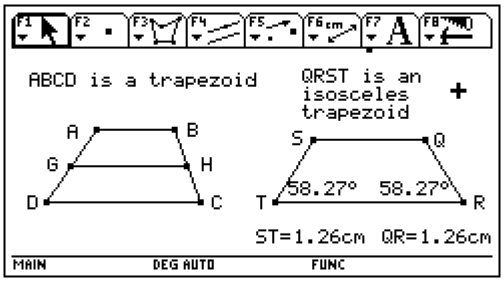

Figure 1

Construct and Investigate:

1. Determine three ways to construct a trapezoid using the Voyage™ 200. Test your constructions by dragging independent points of the trapezoids to be sure that the figures remain trapezoids at all times.
2. Find a relationship regarding angle measurement that appears to be true for all trapezoids. Write a conjecture stating this relationship. Test your conjecture by dragging a vertex of the trapezoid and observing the relationship for a variety of different-shaped trapezoids.
3. Determine three ways to construct an isosceles trapezoid on the Voyage 200. Write a step-by-step procedure for each method, and test these procedures before moving to the next activity.
4. State at least three properties that are true for all isosceles trapezoids. These properties could include statements about angle measures, side lengths, diagonals, area, or perimeter.

Explore:

1. The area of a trapezoid is given by the formula $A = \frac{b_1 + b_2}{2}h$, where b_1 and b_2 are the lengths of the two bases and h is the height of the trapezoid. Drag one of the nonparallel sides of a trapezoid toward the other. Explain how you can derive the area formula for a triangle from the area formula for a trapezoid.
2. Triangles and trapezoids are related in other ways. The **midsegment of a triangle** is a segment connecting the midpoint of any two consecutive sides of a triangle. The **midsegment of a trapezoid** is the segment connecting the midpoints of the two nonparallel sides. Find a relationship between the midsegments of triangles and trapezoids. Investigate the relationship between a midsegment of a triangle or trapezoid and the area. Explain algebraically why this is always true for any triangle or trapezoid.
3. The large trapezoid shown in Figure 2 can be dissected into similar copies of itself. Starting with a dissected trapezoid, use transformational geometry tools to construct the larger trapezoid shown. What properties of this trapezoid allow this transformation? Explain. Can you find other trapezoids that have the same property? Will these figures tessellate the plane? Explain.

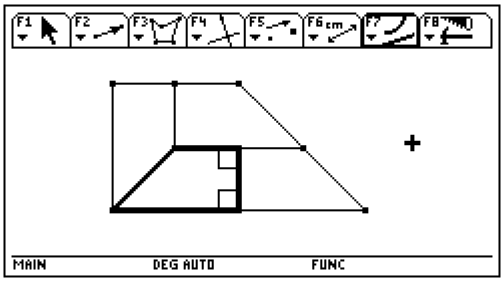

Figure 2

Teacher's Guide: Investigating Properties of Trapezoids

Construct and Investigate:

1. There are many ways to construct a trapezoid. Three ways are listed here, but students may find others that use the properties of trapezoids.
   - Draw two segments connected at a common endpoint. Construct a line parallel to one segment through the noncommon endpoint of the other segment. The fourth vertex of the trapezoid can be any point on the parallel line that is in the interior of the angle formed by the first three points (the vertex being the common endpoint).
   - Construct a triangle, and draw a line parallel to one side passing through any point on one of the other sides. A trapezoid will be formed by the base, the parallel, and the two nonparallel sides. Draw a polygon over this figure, and hide the triangle and the parallel line.
   - Construct a segment, and draw a line parallel to it through a point not on the segment. Connect the endpoints of the segment to any two points on the parallel line so that the nonparallel sides do not cross.
2. As with all convex quadrilaterals, the sum of the interior angles is 360°. The consecutive angles between the parallel sides (bases) of the trapezoid are always supplementary (Figure 3).

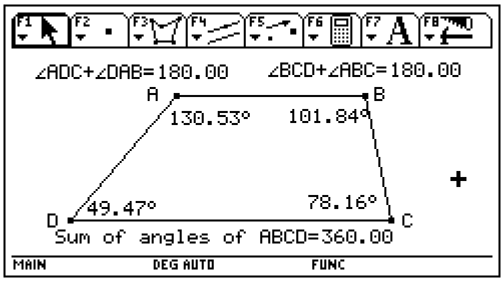

Figure 3

3. Students will find many ways to construct an isosceles trapezoid. Three ways are listed below:
   - Draw a segment, and construct its perpendicular bisector. Pick any point on the bisector other than the midpoint of the segment. Construct segments from the endpoints of the original segment to this point, forming an isosceles triangle (the base being the original segment). Place another point on the perpendicular bisector between the vertex of the isosceles triangle and the midpoint of the segment. Construct a line through this point parallel to the base of the isosceles triangle. Use the **Polygon** tool to form the isosceles trapezoid, using the endpoints of the original segment and the two points of intersection between the parallel line and the sides of the isosceles triangle.
   - Place two points on the screen, and construct any line that does not contain either point. Reflect the two points over the line. Use the Polygon tool to connect the four points to form an isosceles trapezoid.
   - Construct a line segment. Construct the perpendicular bisector of the segment. Select a point on the perpendicular bisector that is not on the segment. Construct a line through this point, parallel to the original segment. Draw a circle with its center at the intersection of the perpendicular bisector and the parallel line. The polygon formed by the endpoints of the original segment and the intersection points of the circle and the parallel line is an isosceles trapezoid.

4. The following properties are true for all isosceles trapezoids:
   - Both pairs of base angles are equal in measure.
   - The perpendicular bisector of one of the bases is also the perpendicular bisector of the other base.
   - The perpendicular bisector of the bases is a symmetry line of the isosceles trapezoid.
   - The diagonals of an isosceles trapezoid are equal in length and divide the trapezoid as follows:
     - Three pairs of congruent triangles (Figure 4).
     - One pair of similar triangles (Figure 5).
   - Isosceles trapezoids are cyclic quadrilaterals, which means the four vertices lie on a circle. Therefore, the product of the diagonals is equal to the sum of the products of the two opposite sides (Ptolemy’s Theorem).
   - The center of the circumscribed circle is the intersection of the perpendicular bisectors of the two nonparallel sides. This center also lies on the line of symmetry of the trapezoid (Figure 6).

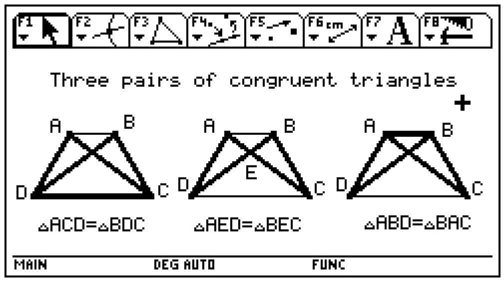

Figure 4

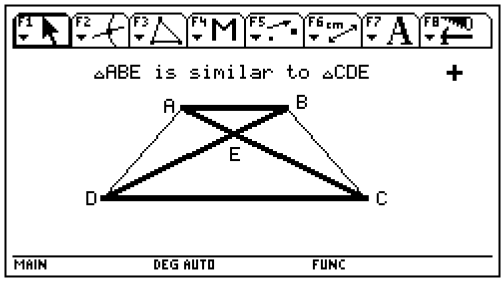

Figure 5

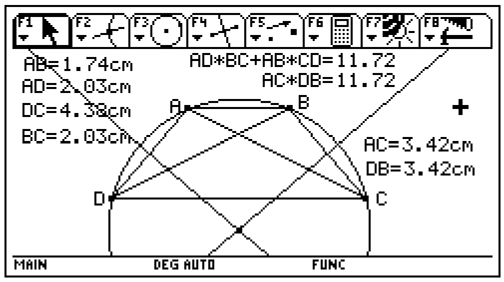

Figure 6

Explore:

1. As the two nonparallel sides of a trapezoid come closer together, the length of the shorter base eventually becomes zero and the trapezoid becomes a triangle (Figures 7 and 8).

   If b_1 represents the shorter base and b_2 represents the longer base, when $b_1 = 0$, the area formula for a trapezoid becomes the area formula for a triangle:

   $$A = ½\,(0 + b_2)\,h = ½\,bh.$$

   In a sense, a triangle is a special case of a trapezoid where the shorter base has a length of zero.

2. The length of the midsegment of a trapezoid is equal to the average length of the two parallel sides (Figure 9). The midsegment of a triangle is equal to half the length of its parallel base. If a triangle is considered a special case of a trapezoid with one base zero in length, then the same relationship holds for both figures.

   The area of a trapezoid is the product of the length of its midsegment and its height (Figure 10). The same is true for a triangle.

   If b_3 represents the length of the midsegment of a trapezoid and you know that $b_3 = \frac{b_1+b_2}{2}$, and if the area of a trapezoid is $A = ½\,(b_1 + b_2)\,h$, then by substitution, you know that the area of a trapezoid is given by $A = b_3h$.

   In a triangle, the midsegment $b_3 = \frac{b_2}{2}$.

   If the area is given by $A = ½\,bh$, then you know that $A = b_3h$.

   In both cases, the area of the figure is given by the product of the length of its midsegment and its height.

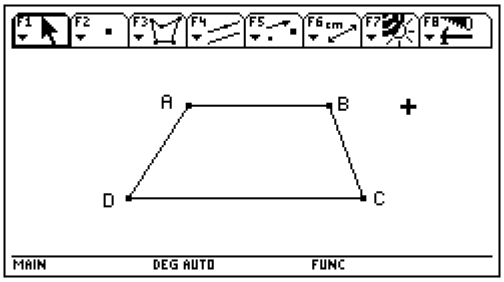

Figure 7

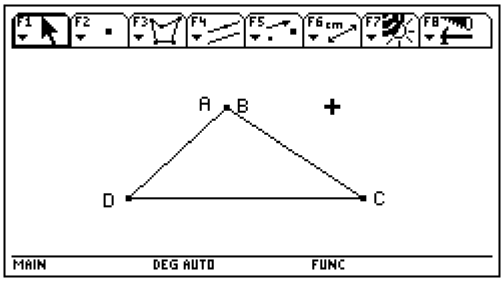

Figure 8

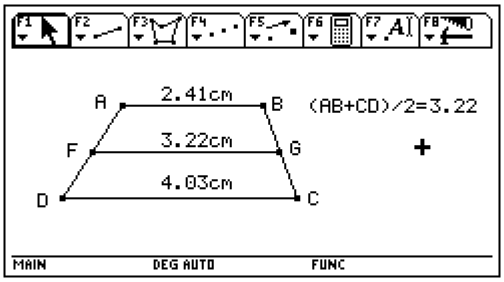

Figure 9

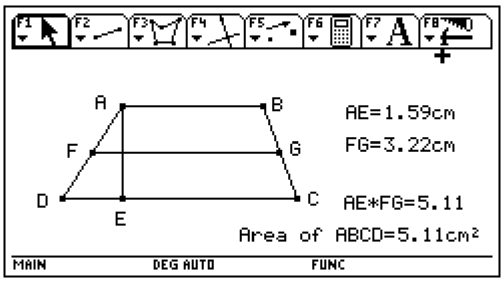

Figure 10

3. Each of the four subparts of the whole trapezoid are congruent and similar to the entire figure with a side ratio of 2 and an area ratio of 4.

   Each trapezoid is also special in the following ways:

   - The two bases make right angles to one nonparallel side.
   - The other nonparallel side makes an angle of 45° to one base and 135° to the other base.
   - The longer base is twice as long as the shorter base (Figure 11).

   The construction can be made using reflections and rotations or using reflections and translations.

   Other trapezoids of this type are shown in the figures on the right.

   - Figure 12 is a trapezoid formed by a square and half of a congruent square.
   - Figures 13 and 15 are trapezoids formed by three congruent equilateral triangles so that the angles of the trapezoid are either 60° or 120°.
   - Figure 14 is made up of trapezoids formed by two congruent squares and a 30°-60°-90° triangle.

   These figures are fun to do and can involve many creative and imaginative constructions. Once they determine the original trapezoid, see whether your students can construct each one using only isometries.

   Because each example can be dissected or expanded into an infinite number of identical parts, all of them can tessellate the plane.

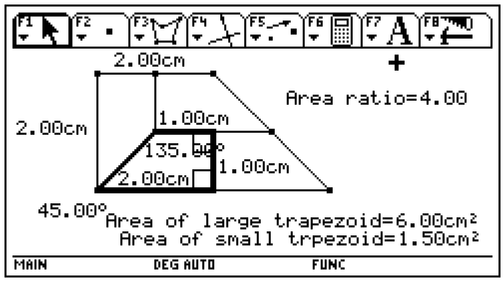

Figure 11

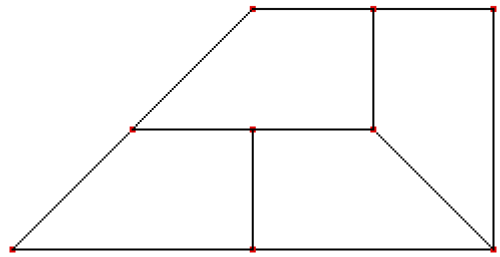

Figure 12

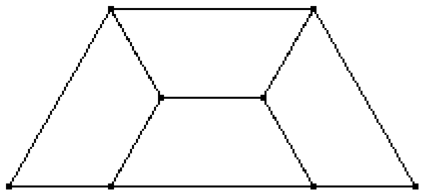

Figure 13

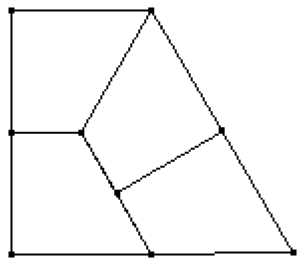

Figure 14

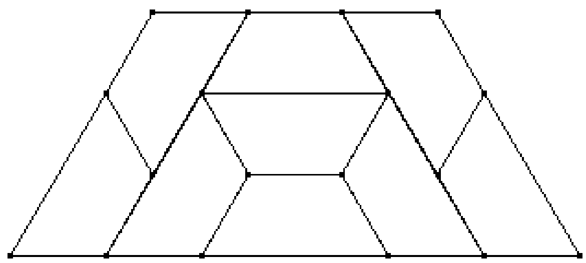

Figure 15

Investigating Properties of the Diagonals of Quadrilaterals

Definitions:

- **Quadrilateral—**a four-sided polygon.
- **Diagonal of a quadrilateral**—a segment that connects opposite vertices.
- **Kite**—a quadrilateral with two distinct pairs of consecutive sides congruent.
- **Trapezoid—**a quadrilateral with at least one pair of parallel sides.
- **Isosceles trapezoid**—a trapezoid with one pair of base angles congruent.
- **Parallelogram**—a quadrilateral with two pairs of sides parallel.
- **Rectangle**—a quadrilateral with four right angles.
- **Rhombus**—a quadrilateral with four congruent sides.
- **Square—**a quadrilateral with four congruent sides and four right angles.

Construct and Investigate:

1. Figure 1 shows a quadrilateral hierarchy in which each quadrilateral shares all the properties of those connected above it in the chart. Construct each figure in the chart, and verify that the definitions allow each type of quadrilateral to be a special case of the quadrilaterals connected above it in the chart.
2. For each figure, write as many conjectures as possible about its diagonals. Be sure that the statements are true for the figure and all connected below it but not true for the figures connected above it in the chart.

   For example, the diagonals of an isosceles trapezoid are congruent. This is also true for rectangles and squares but not for trapezoids or quadrilaterals.

   Each property should appear only once in your list, associated with the figure that first displays the property in the hierarchy chart.

   Use the Voyage™ 200 with Cabri to assist in the investigations. Examine relationships involving lengths, angle measures, symmetry, bisection, collinearity, areas, and whatever else comes to mind.

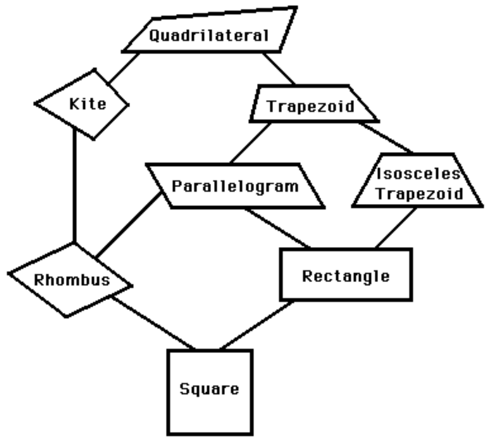

Figure 1

Explore:

1. Investigate the relationship between the four triangles formed by the diagonals of any convex quadrilateral. See whether you can discover a relationship that is always true by doing some creative explorations and measurements.
2. Carpenters and other builders use a property of the diagonals of a rectangle to make sure that they have four right angles when they build a wall or square up the foundation of a house. See whether you can discover which property they use. Check with someone in the building trades in your community to verify that this property is actually used. Ask your contact person to share other ways in which geometry is used in construction.

Teacher's Guide: Investigating Properties of the Diagonals of Quadrilaterals

Construct and Investigate:

The definitions and hierarchy chart in this activity are from *Geometry* (Coxford, Usiskin, and Hirschhorn, 1991). These definitions may not be entirely consistent with other textbooks. In particular, quadrilaterals and kites are not necessarily convex, and trapezoids are defined as having at least one pair of parallel sides. In this activity quadrilaterals are defined as polygons, and therefore, crossed figures with four sides are not considered quadrilaterals. You may want students to use a different set of definitions than those presented here. You should be aware, however, that such definitions will change some of the statements and/or their location in the hierarchy listed below.

This activity takes considerable time to complete if each student or group of students does the entire exploration. An alternative technique may be to assign different groups different quadrilaterals to construct and investigate. The groups could then report their findings and assemble a composite list. Resolve conflicts with either large group discussion, or small student groups. You can assign certain quadrilaterals to particular groups or use a random selection process. The properties of some quadrilaterals appear more difficult to find than others.

There are many ways to construct a particular quadrilateral besides using the definition. It may be worthwhile to have students determine different constructions based on properties of various quadrilaterals. For more information about kites and trapezoids, see the Teacher's Guides for the activities *Investigating Properties of Kites* and *Investigating Properties of Trapezoids*.

The properties of diagonals listed for each quadrilateral are valid for that figure and for all quadrilaterals connected below it in the hierarchy chart (Figure 1). These properties include:

Quadrilaterals

- The quadrilateral has two diagonals.
- The sum of the lengths of the diagonals is less than the perimeter of the quadrilateral.
- The sum of the lengths of the two diagonals is equal to the perimeter of the quadrilateral formed by connecting the midpoints of the sides of the original quadrilateral.

Kites

- The diagonals of the quadrilateral are perpendicular.
- At least one diagonal of the quadrilateral is the bisector of the other.
- At least one diagonal bisects two angles of the quadrilateral.
- At least one diagonal lies on a symmetry line of the quadrilateral.
- One-half the product of the diagonals is equal to the area of the quadrilateral.

Trapezoids

- The diagonals of the quadrilateral intersect. (This property is true for all convex quadrilaterals, but this is the first level in the hierarchy where you are assured that the quadrilateral is convex.)
- The products of the areas are equal for opposite pairs of triangles formed by the intersection of the diagonals of a quadrilateral. (This property is true for all convex quadrilaterals, but this is the first level in the hierarchy at which you are assured that the quadrilateral is convex.)
- The angles formed between a diagonal of the quadrilateral and its bases are congruent.
- The diagonals divide the quadrilateral into four triangles such that the areas are equal for at least one pair of these triangles (Figure 2).
- The intersection point of the diagonals is collinear with the midpoints of the bases of the quadrilateral (Figure 3).
- For both diagonals of the quadrilateral, the ratio of the distances from the midpoints of the bases to the intersection point of the diagonals is the same as the ratio of the distances from the vertices to the diagonal intersection point.

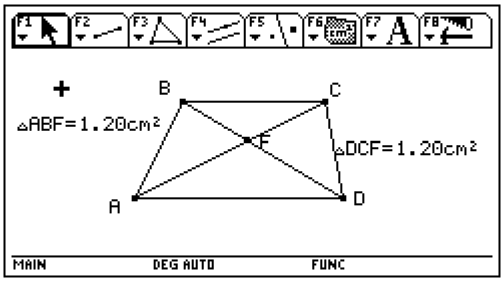

Figure 2

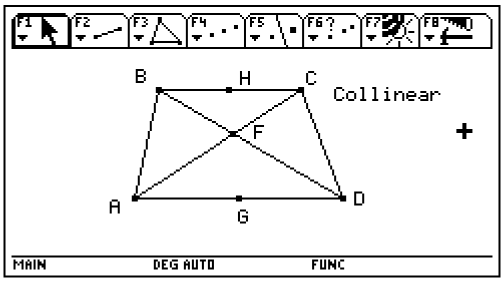

Figure 3

Isosceles Trapezoids

- The diagonals of the quadrilateral are congruent.
- The diagonals divide the quadrilateral into three pairs of congruent triangles and one pair of similar isosceles triangles.
- At least one symmetry line of the quadrilateral contains the intersection point of the diagonals.

Parallelograms

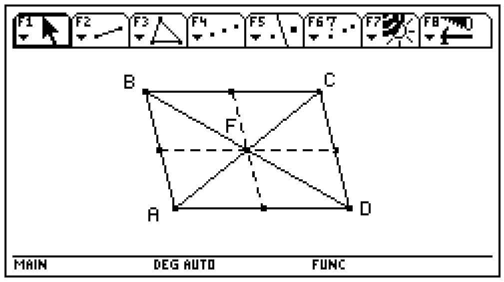

Figure 4

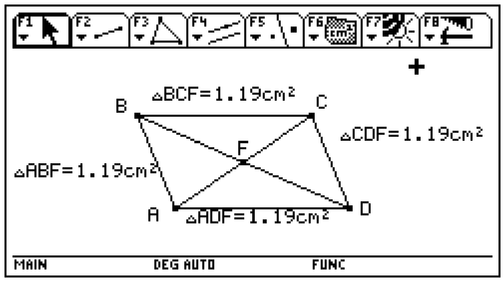

Figure 5

- Each diagonal divides the quadrilateral into two congruent triangles.
- The diagonals of the quadrilateral bisect each other.
- The diagonals divide the quadrilateral into two pairs of congruent triangles.
- The intersection point of the diagonals is also the intersection point of the lines connecting the midpoints of the opposite sides of the quadrilateral (Figure 4).
- The diagonals meet at the centroid of the quadrilateral (Figure 4).
- The intersection point of the diagonals is a bisector of the segment connecting the intersection points of the perpendicular bisectors of the sides of the quadrilateral (Figure 4).
- The diagonals divide the quadrilateral into four triangles of equal area (Figure 5).

Rectangles

- The diagonals of the quadrilateral are congruent and bisect each other.
- The intersection point of the diagonals is also the intersection point of the perpendicular bisectors of the sides of the quadrilateral.
- The length of each diagonal is equal to the square root of the sum of the squares of two consecutive sides of the quadrilateral.

Rhombi

- Both diagonals bisect opposite angles of the quadrilateral.
- Both diagonals form symmetry lines for the quadrilateral.
- The diagonals divide the quadrilateral into four congruent right triangles.
- The intersection point of the diagonals is the incenter of the quadrilateral.
- The diagonals lie on the symmetry lines of the quadrilateral.

Teacher's Guide: Investigating Properties of the Diagonals of Quadrilaterals (Cont.)

Squares

- The diagonals divide the quadrilateral into four congruent isosceles right triangles.
- The intersection point of the diagonals is the circumcenter of the quadrilateral.
- The diagonals meet at the symmetry point of the quadrilateral.
- Any line through the intersection of the diagonals divides the perimeter and the area of the quadrilateral into two equal parts.
- The ratio of the length of a diagonal to a side is equal to $\sqrt{2}$.

Explore:

1. The products of the areas of opposite triangles formed by the diagonals of a convex quadrilateral are always equal (Figure 6).
2. When laying out the framing for a wall or the foundation of a rectangular home, builders measure the diagonals of the parallelogram and adjust the structure until the lengths of the diagonals are congruent. This assures that the four corners are square and the wall or foundation is rectangular (Figure 7).

   Another technique for squaring a wall uses the properties of a right triangle with legs of 3 and 4 units and a hypotenuse of 5 units (or multiples of these values).

   A carpenter's square, or framing square, is a tool used to cut rafters, steps, and braces as well as make right-angle cuts. This tool uses the properties of right triangles using trigonometry and the **Pythagorean Theorem**.

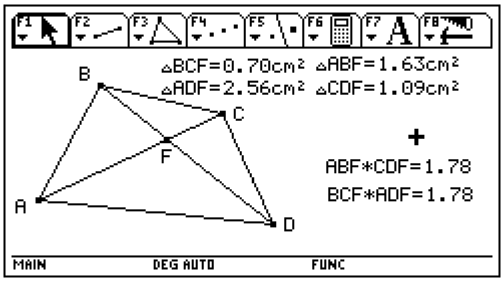

Figure 6

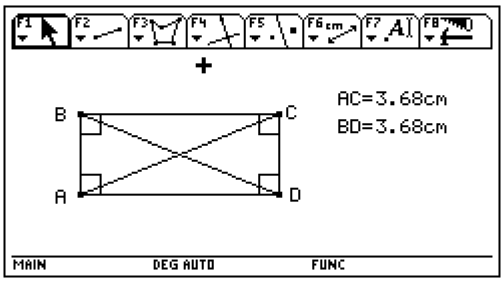

Figure 7

The Orthocenter of a Triangle

Definitions:

Orthocenter—the point of concurrence of the three altitudes of a triangle. Figure 1 shows othocenter ***H*** and the nine-point circle.

Nine-point circle, or **Feuerbach circle**, of a triangle—the circle centered at the midpoint between the orthocenter and the circumcenter of the triangle that passes through the midpoints of the sides (three points), the feet of the altitudes (three points), and the midpoints of the segments connecting the vertices to the orthocenter (three points).

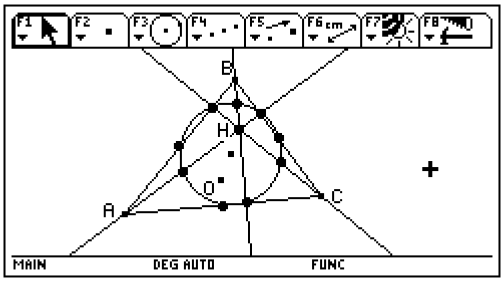

Figure 1

Construct and Investigate:

1. Draw and label $\triangle ABC$ on the Voyage™ 200. Construct the altitudes, and locate the orthocenter of the triangle. Would you need to construct all three altitudes? Explain why or why not.
2. Hide the altitudes, and drag the triangle around the Voyage 200 screen. Explain where the orthocenter is located for different types of triangles.
3. Hide the original triangle so that only the three points ***A***, ***B***, and ***C*** (representing the vertices), and the orthocenter ***H*** remain on the screen. Drag the vertices around the screen, and investigate the relationship of these four points to one another. Test your conjectures.
4. How many ways can three items be chosen from a collection of four items if order is not important and there is no replacement? Choose all the subsets of three of the four points on the screen, and construct all possible triangles defined by these four points. Draw the nine-point circle for each of the triangles you constructed. (**Hint:** Make a macro after the first one is constructed.) Drag the original triangle around the screen. Explain what happens. (**Hint:** Point at the nine-point circle with the **Pointer** arrow. What happens?)
5. Using a macro, construct the circumcircles of the triangles in your figure. What is the relationship of these circles to each other? Explain. What is the relationship between these circumcircles and the nine-point circle? Explain.
6. Hide the circumcircles and the nine-point circles, and connect the circumcenters of the four triangles in this figure. Investigate the relationships between the original figure and the one formed by connecting the circumcenters. Explain the relationships you find.
7. Construct the nine-point circle for the new figure formed in part 6 above. How does this compare to the nine-point circle for the original figure? Explain.

Explore:

1. Draw and label a new acute triangle, and construct its orthocenter with three altitudes. Draw the **orthic triangle**, the triangle with vertices located at the feet of the altitudes of the original triangle. Explore the relationships between the original triangle and the orthic triangle by investigating the relationships between their incenters, incircles, circumcenters, circumcircles, orthocenters, and nine-point circles. Explain what you find and how all the different centers and circles are related.
2. The lines representing the sides of a bisected angle are called the **isogonal lines**, and each one is called the **isogonal conjugate** of the other. By this definition, adjacent sides of a triangle are isogonal conjugates of each other. Investigate the angular relationship between the orthocenter, the incenter, the circumcenter, and a vertex of a triangle. Explain what you find.

Teacher's Guide: The Orthocenter of a Triangle

Construct and Investigate:

1. The altitudes of a triangle intersect at the orthocenter. The intersection of any two of the three altitudes locates the orthocenter because two nonparallel lines in a plane determine a unique point of intersection. It is interesting that all three altitudes are coincident at the same point, but once this is established, only two altitudes are needed to locate the orthocenter.
2. When the triangle is acute, the orthocenter is located in the interior of the triangle because the feet of the altitudes lie on the sides of the triangle. When the triangle is obtuse, the orthocenter is exterior to the triangle because two of the altitudes fall outside the triangle. When the triangle is a right triangle, the orthocenter is located at the vertex of the right angle because two of the altitudes of a right triangle are the legs of the right angle.
3. One of the most beautiful symmetries of a triangle is represented by the relationship of the **orthic set of points** made up of the vertices of a triangle and its orthocenter. The triangle formed by any combination of three of these points has the fourth point as its orthocenter (Figure 2).

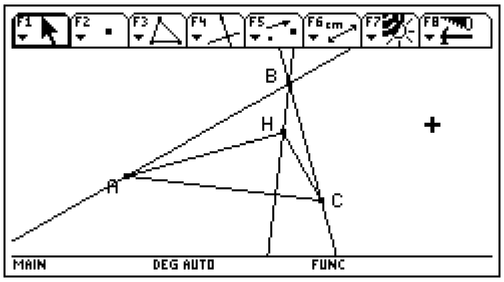

Figure 2

4. The **combination** of four items taken three at a time is given by the equation:

   $_4C_3 = \frac{4*3*2}{3*2*1} = 4.$

   Four triangles are possible using this set of four points. The nine-point circles for all four triangles are the same (Figure 3). By definition, the nine-point circle of a triangle passes through the feet of the altitudes, the midpoints of the sides, and the midpoints of the segments joining the vertices to the orthocenter of the triangle. Because each point in the orthic set is the orthocenter of the triangle formed by the other three, the sides of each triangle lie on the altitude of another triangle. The midpoint of each side of a triangle is also the midpoint between a vertex and an orthocenter. Because the same nine points are interchangeable, all four triangles have the same nine-point circle.

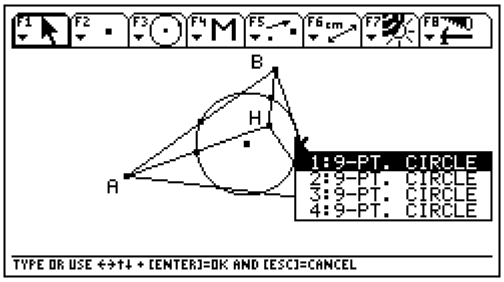

Figure 3

   An interesting extension of this idea is to find the **centroids** of the four triangles formed by the orthic set of points. These centroids also form an orthic set with all the same properties of the orthocenters (Figure 4).

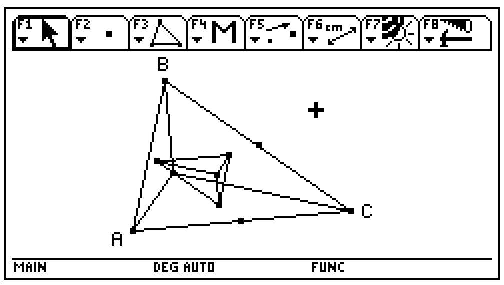

Figure 4

The figure formed by the orthic set of centroids is similar to the original figure, but the segments are only one-third the length of the corresponding segments in the original figure; the corresponding areas are only one-ninth as large as the original (Figure 5).

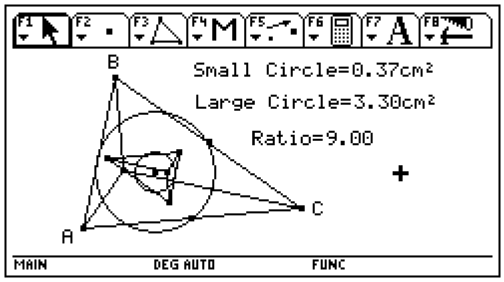

Figure 5

This area comparison includes the corresponding circumcircles and nine-point circles. The nine-point circles of two figures are concentric. This means that the original figure can be transformed onto the smaller figure by a rotation about the center of the nine-point circle, followed by a one-third dilation.

5. The areas of the four circumcircles are equal; hence, the circumcircles are congruent (Figure 6).

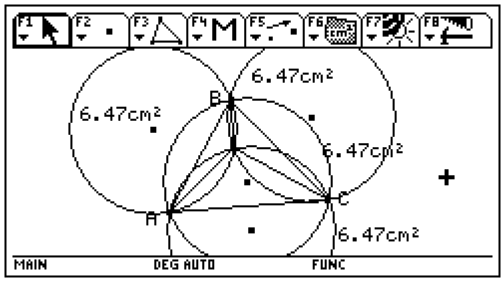

Figure 6

The area of the nine-point circle is one-fourth the area of any one of the circumcircles. This is a property of any nine-point circle when compared with its corresponding circumcircle (Figure 7).

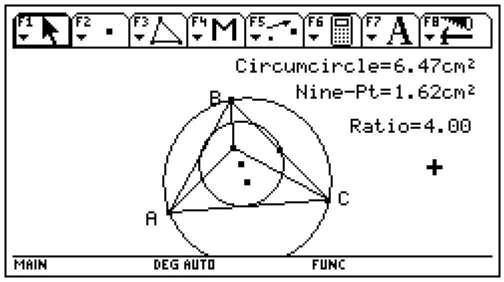

Figure 7

6. When all of the circumcenters are connected to all the other circumcenters, a figure congruent to the original figure is formed (Figure 8).

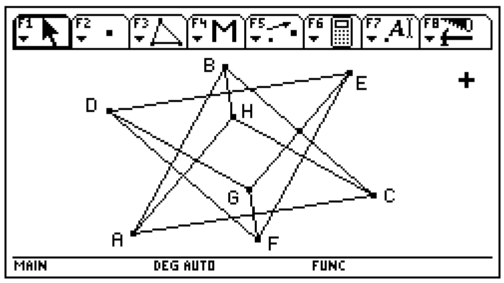

Figure 8

The circumcenters of the original figure, ***D***, ***E***, ***F***, and ***G***, have become the orthocenters of the new figure. This means that the orthocenters of the original figure, ***A***, ***B***, ***C***, and ***H***, are the circumcenters of the new figure.

7. Any triangle in the new figure has the same nine-point circle as any triangle in the original figure (Figure 9).

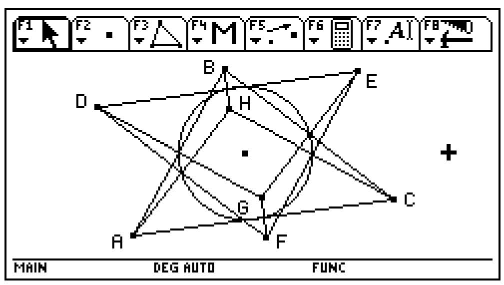

Figure 9

The center of the nine-point circle is the midpoint of the segment connecting the orthocenter to the circumcenter. Because the orthocenters and circumcenters of these two figures are interchangeable, the center of the nine-point circle represents the center of rotation that transforms the original figure into the new figure.

Explore:

1. Figure 10 shows $\triangle ABC$ with orthocenter H and **orthic triangle** $\triangle DEF$. The orthic triangle represents the inscribed triangle with the smallest perimeter.

   For more information on this property, see "Fagnano's Problem" in *Explorations for the Mathematics Classroom* (Vonder Embse and Engebretsen, 1994, p. 36).

   The angle bisectors of the orthic triangle are the altitudes of $\triangle ABC$; therefore, the orthocenter H is also the incenter of $\triangle DEF$ (Figure 10).

   The nine-point circle of $\triangle ABC$ is the circumcircle of the orthic triangle $\triangle DEF$. This nine-point circle is four times the area of the nine-point circle of $\triangle DEF$ and one-fourth the area of the circumcircle of $\triangle ABC$ (Figure 11).

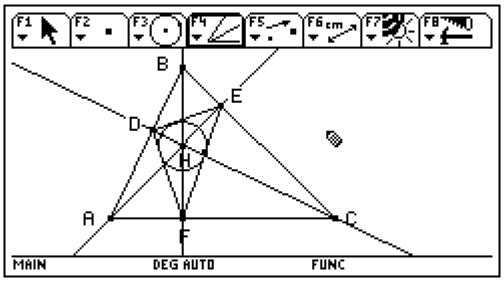

Figure 10

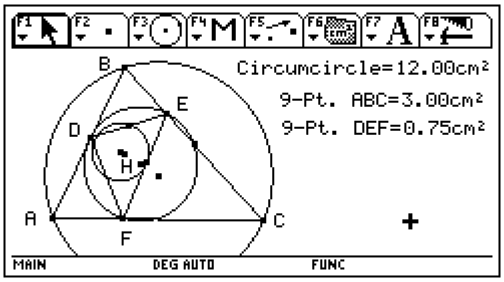

Figure 11

2. For $\triangle ABC$ shown in Figure 12, $\triangle HBO$ is formed by the circumcenter O, a vertex B, and the orthocenter H. It is bisected by the line through the vertex B and the incenter I.

   In Figure 12, segments $\overline{HB}$ and $\overline{OB}$ are isogonal conjugates of each other because line BI bisects $\triangle HBO$. This relationship is true for each vertex of the triangle.

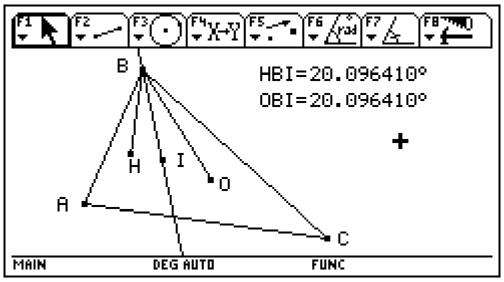

Figure 12