Dynamic Algebra

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Cabri Geometer and Geometer Sketchpad showed us the astonishing effects that can be produced by dynamical change of the base objects. A classical problem leads to new perceptions and visual thinking (e.g. the way an ellipse can be transformed via a parabola to a hyperbola). There is a lack of interesting problems for the use of CAS-calculators (CAS = computer algebra system). Parameterisation is an easy way to create attractive and challenging problems derived from classical ones. Such problems will lead students to a new way of thinking, similar to the geometrical approach described above.
I would like to demonstrate the method with the following examples. Examples of this kind can be found in any classical exercise book.

Classical problem 1: Fractional equation

Solve: $\frac{x}{2x-8} + \frac{x-6}{x-4} = \frac{3}{2}$

Store the equation in eq and solve it with solve(eq,x)!

Result: TRUE. All x∈|R are solutions, or nearly all. The CAS-calculator doesn't check the domain of the equation.

Dynamization of the classical problem

a) Replace 4 with a! Give an interpretation of the result!
The result is the Boolean expression x=6 or a=4 (means x=6 or a=4 and x arbitrary). Without first solving the classical problem, students can hardly understand it. The interpretation of such expressions helps to improve logical thinking.

b) In addition replace in a) furthermore 8 with b!
The result x = $\frac{3b(a-4)}{b+4(a-6)}$ seems to be incomplete and leads to a discussion of the reliability of the solver. Indeed for a=4 the solution is 0 independent of b (compare with a)). This is a nice motivation to investigate the situation without technology.
CAS-calculators often give incomplete or even false solutions, especially with fractional equations.

c) Replace in the classical problem $\frac{3}{2}$ with 3!
Result: FALSE.
If you solve the equation by hand you get x=4. In this case the calculator considered the domain of definition of the equation.

d) Replace in c) 4 with a as in problem a)!
The result x = $\frac{5a+4 \pm \sqrt{25a^2 - 248a + 592}}{6}$ is impressive and would be a hard piece of work without technology.
Further questions are possible: For what a will we have one, two or no solution(s)?
Result: $25a^2-248a+592=0 \Leftrightarrow \quad a = \frac{148}{25}$ or a=4. As we know from c) a=4 isn't a solution.
Once more the solver arrives at an incomplete result.

Classical problem 2: Heron’s formula

(From *Richard G. Brown: Advanced Mathematics*, page 6, exercise 32)

Find the area of the triangle with vertices A=(-13/2), B=(5/17) and C=(22/-4), using Heron’s formula.

Solution:

Define Heron’s formula for the area of a triangle as a function:
$\sqrt{s*(s-a)*(s-b)*(s-c)} \;\; | \;\; s=\frac{a+b+c}{2}$ –>heron(a,b,c)
Test the Formula with a=3, b=4, c=5!
Store the vertices: [-13,2]–>a, [5,17}–>b, [22, -4]–>c
heron(norm(b-a),norm(c-b), norm(a-c)) Result: 316.5
Note: The variables a, b, c and even s are local variables. They are not overwritten by the definition of the vertices.
The above exercise doesn’t require a CAS-calculator. You can also do it with a small program.

Dynamization of the classical problem

a) Is it possible to find a triangle with sides x, x+1, x+2 (arithmetical sequence) and area 10? (Result: x=4.018 ~~or -6.018~~)
b) What is the biggest (smallest) area, you can replace 10 in a), to get a value of *x*? (Individual exploration!). Result: It works for all areas!
c) A heronic triangle is a triangle whose sides and areas are natural numbers. Find all heronic triangles whose sides are smaller than 20.
Solution: Write a program which calculates the possible triangles.

You can imagine a lot of other questions, e.g. about isosceles or equilateral triangles or triangles with a given ratio of its sides.

Classical problem 3: Vector equation

An aeroplane holds course east with an unknown effective velocity of *e*. The wind blows direction north-east with w=100 miles/h. The aeroplane is flying with a velocity of f=500 miles/h.
What’s the maximal effective velocity of the plane and which course should the pilot hold (flight direction vector $\vec{f}$ = [x,y])?

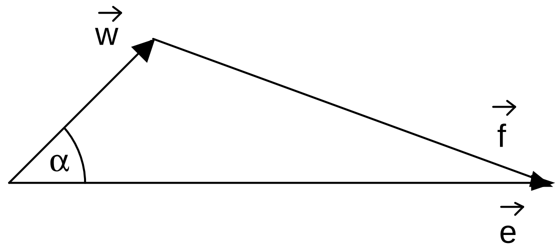
Solution:

You can take either a geometrical or an algebraic approach as in the following solution:

$\frac{100}{\sqrt{2}}$ [1,1] + [x,y] = e[1,0] –> e1

$x^2 + y^2 = 500^2$ –> e2

solve(e1[1,1] and e1[1,2] and e2, {x,y,e})



Result: $x = 350\sqrt{2}$, $y = -50\sqrt{2}$, $e = 400\sqrt{2} = 565.685$ (and the second “solution” in the opposite direction $x = -350\sqrt{2}$, $y = -50\sqrt{2}$, $e = -300\sqrt{2}$)

Dynamization of the classical problem

a) The velocity of the wind and the plane will now be replaced with parameters w and f. As the wind force changes, the direction vector of the plane has to change too in order to hold a constant effective velocity. The formulas allow an automatic control of the flight.

Solution: Replace the two equations with

$\frac{w}{\sqrt{2}}$ [1,1] + [x,y] = e[1,0] –> e1 $x^2 + y^2 = f^2$ –> e2

Result: $x=\frac{\sqrt{2(2f^2-w^2)}}{2}$, $y=\frac{-\sqrt{2}w}{2}$, $e=\frac{\sqrt{2}(\sqrt{2f^2-w^2}+w}{2}$

You can ask questions like: Which is the minimal flight velocity, allowing the pilot to hold the eastern course.

b) A further generalisation is to replace the wind direction vector [1,1] with [a,b]. (If you want, you can also describe *a*, *b* with the wind direction angle.)

Solution: [a,b] + [x,y] = e[1,0] –> e1, (e2 stays unchanged), $a^2 + b^2 = w^2$ –> e3
Solve(e1[1,1] and e2[1,2] and e2 and e3,{x,y,e})

Result: Students have to analyse the solutions and choose the most appropriate one, which should improve their abstract thinking ability. The solution is:

$x=\frac{\sqrt{(a^2+b^2)f^2-b^2w^2}}{\sqrt{a^2+b^2}}$ and $y=\frac{-bw}{\sqrt{a^2+b^2}}$ and $e=\frac{\sqrt{(a^2+b^2)f^2-b^2w^2}+aw}{\sqrt{a^2+b^2}}$

Caution: The result depends on the mood of the calculator. It can also be given in the form:

$x=\sqrt{f^2-b^2}$ and y=-b and $a^2+b^2=w^2$ and $l=a+\sqrt{f^2-b^2}$

Students are now motivated to start a flight simulation project (at least in an ideal world).

Classical problem 4: Transversal

Given two skewed straight lines *g* and *h*. Insert a horizontal transversal of length 5! At which height do you have to place the transversal?

g: (x/y/z) = (-8/5/7) + t (-6/3/4) h: (x/y/z) = (0/1/-5) + s (1/2/4)

Solution:

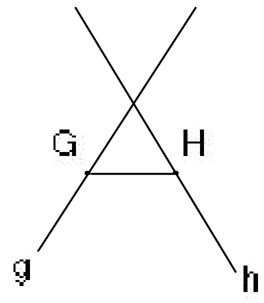
Store the above vector equations of the lines in *g* and *h*!

The problem can be expressed by two conditions:

I. $|\overrightarrow{GH}| = 5$ (CAS: norm(g-h) = 5 –>e1)

II. The 3rd component of $\overrightarrow{GH}$ is 0 (CAS: (g-h)[1,3] = 0 –>e2)

The solutions to the system of equations I and II are
(t/s) =(-1/2) and (-2/1).
The calculation with paper and pencil is rather hard work. However it requires only basic algebraic manipulations, which reduce the problem to a quadratic equation. With the calculator it can be done by pressing a few keys (solve(e1 and e2,{s,t})).
Next insert the t- or s-value in the 3^{rd} component of g or h.
(CAS: g[1,3] | t=-1 and g[1.3] | t=-2)



Dynamization of the classical problem

The above problem can be generalized using the same equations but replacing values with parameters, e.g.:

a) Replace the length 5 in condition I with a parameter *c*. What is the shortest horizontal connection between the straight lines *g* and *h*?
Solution: $c=\frac{5\sqrt{2}}{2}$

b) Omit condition II in the classical problem! The transversal of length 5 glides on the two straight lines. What is the possible range of the variables *t* and *s* for a solution?
Solution: The extreme values are $t=\frac{5(-12\pm\sqrt{21})}{41}$ and $s=\frac{48\pm5\sqrt{61}}{41}$

c) Omit condition II again, but replace 5 with *c* (as in a)). Determine the endpoints and the length of the shortest distance between *g* and *h*! Can you arrive at the solution in another way considering problem b)?
Solution: Shortest distance $c=\frac{20\sqrt{41}}{41}$ for $t=-\frac{300}{205}=-\frac{60}{41}$ and $s=\frac{240}{205}=\frac{48}{41}$
This is just the mean value of the extrem values of the solutions found in b).
Now put s and t in the vector equations of g and h!

d) Replace in the classical problem one component of the first vector in *g* by a parameter *c* (e.g. *c* instead of 7). Determine the value of *c*, for which *g* and *h* intersect!
Solution: solve((g-h)[1,1]=0 and (g-h)[1,2]=0,{s,t}) Result: s=0 and t=-4/3
(g-h)[1,3]=0|s=0 and t=-4/3 gives c=1/3

e*) Project for interested students: In problem b) vary parameter t within the possible range. Describe the other endpoint of the gliding transversal using the same parameter (two possibilities). Visualize the gliding transversal geometrically (by programming or animation in a geometry program).

Classical problem 5: Apollonius sphere

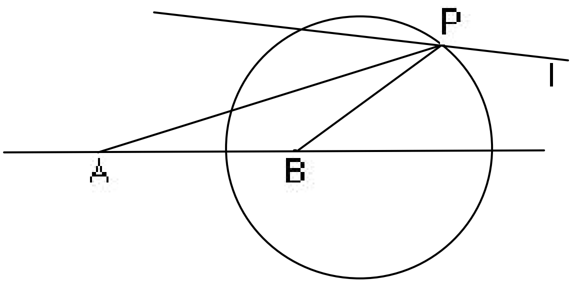
Given the points A=(4/-5/11), B=(1/$-\frac{1}{2}$/2) and the line *l*: (x/y/z) = (0/–13/0) + t (3/8/1).

Find the point P on the line *l*, so that |AP| : |BP| = 2:1 !

Solution:
The problem can be expressed by two conditions:

I. $\frac{|\vec{AP}|}{|\vec{BP}|}=2$ II. $P \in l$

First store the points A and B in *a* and *b* and the line in *l*
Solve equation I: solve(norm(l-a)/norm(l-b) = 2,t). (result t=1 or t=2)
Substitute *t* with the values found: *l* | t=1, respectively *l* |t=2
Solutions: $P_1 = (3/–5/1)$, $P_2 = (6/3/2)$



Dynamization of the classical problem

The above problem can be generalized using the same equations but replacing values with parameters, e.g.:

a) Replace *l* with *c* in the direction vector of the line *l*.
For which parameters *c* will you find one, two or no solution(s)? Give a geometrical interpretation!

Result: Solution of the equation I: $t_{1,2} = \frac{c-112 \pm \sqrt{-147c^2-224c+17}}{c^2+73}$

One solution: c=2.759 or c=-4.290
Two solutions: -4.290<c<2.759
No solution: c<-4.290 or c>2.759

b) Replace in the line *l* of a) the vector (0/-13/0) with (0/1/-1)!
Give a geometrical interpretation!

Result: $t_{1.2} = \frac{\pm 7}{\sqrt{c^2+73}}$ (2 solutions, symmetrical to the point (0/1/-1))

Interpretation: (0/1/-1) must be the center of the sphere.

c) Replace in the original classical problem the vector (0/-13/0) with (0/30/c).
Result: No solution! (The set of lines given by *c* never intersects the circle.)

d) Let us omit condition II in the original problem
Result: Equation of the Apollonius sphere $x^2+(y-1)^2+(z+1)^2=49$ (compare with problem b)

You can also introduce parameters in A and B and make investigations on the harmonic points A, B and the point of intersection of the Apollonius sphere with *l*

Classical problem 6: Center of a circle

Find the center of a circle with peripheral points A=(-3/1), B=(1/-1), C=(3/-2).
Find also the radius r.
The first part of the problem can be expressed by the conditions

I. |AM| = |BM| II. |BM| = |CM|

With M given, the radius is r = |AM|.

Solution:

First store the center M = [x,y] and the points A, B and C in correspondent variables:

[x,y]–>m [-3,1]–>a [1,-1]–>b [3,-2]–>c

Then define the system of equations I and II and solve it:

norm(m-a) = norm(m-b) –> e1
norm(m-a) = norm(m-c) –> e2
solve(e1 and e2,{x,y})

The result for x is a very large number (6.10543E6) and the calculator gives the warning: "More solutions may exist". The equations e1 and e2 have roots, which can be eliminated by squaring:

solve(e1^2 and e2^2,{x,y})

Now we get the correct result "FALSE", which means that there is no intersection point. The points are on a straight line.

Replacing -2 with -2.1 in C leads to the intersection point

$x = -\dfrac{1581}{40}$ and $y = -\dfrac{1541}{20}$

Although this result is correct, the calculator gives the warning: "May introduce false solutions."

Dynamization

a) Replace -2 in C with a constant *k* and calculate the center of the circle again!

Result: $x = \dfrac{k^2 - 4k + 3}{4(k+2)}$ and $y = \dfrac{k^2 + 11}{2(k+2)}$

The case k=-2 is not dealt with by the calculator.
Note: In this problem it would probably be better to start with C=(3/k) and ask students to analyse special cases including geometrical interpretation.

b) Simplify equations e1 and e2 with

left(e1^2)-right(e2^2) = 0 (analogous with e2)

and you get the linear equations of the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$.

c) Introduce a factor f in equation I): |AM| = f |AC|. The equation I) represents an Apollonius circle as in the problem above.
Result in simplified form: $(1-f^2)x^2 + (2f^2+6)x + (1-f^2)y^2 + (-2f^2-2)y - 2f^2 + 10 = 0$

It is a rewarding exercise to transform this equation to the normal form (with or without technology):

$$\left(x + \frac{f^2+3}{1-f^2}\right)^2 + \left(y - \frac{f^2+1}{1-f^2}\right) = \frac{20f^2}{(f^2-1)^2} = (\text{radius})^2$$

You can ask questions like: What happens with the center and the radius, if f tends to *1* or 0! Describe a straight line as limit of a circle.

You can also generalize the classical problem three dimensionally: Given four points A, B, C and D. Calculate the center M and radius of the sphere through A, B, C, D.

Final note:

This method of dynamization by parametrization works with about one third of all classical mathematical problems and gives us a quick way to produce problems for CAS calculators.

The method is also a way to progress from skill-oriented problems to thinking-oriented problems.

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