Introduction to GPS I - Two Dimensional Preliminaries

NOTE: While working with this lesson, we recommend that you open a 2D-Plot Window and then choose the option Window > Tile > Vertically. Later in this lesson, you will be asked to modify the range of the window using the Plot Window options Set > Plot Range. Using this screen format, you will be able to follow the lesson and view the corresponding graphics displays that you will generate.

Acknowledgement: This activity was inspired by a talk given by Mr. Ralph Irons of Central Shenandoah Valley Regional Governor's School at the 2001 T^3 International Conference in Colombus, OH. His presentation can be found in the Proceedings of that Confence call "2001 a Technological Odyssey." The presentation given here is distinctly different from that of Mr. Irons, but his excitement and exposition were the inspiration.

Background - Step 1:

The classical definition of a circle is: *the locus of all points equidistant from a fixed point.* The measure of the distance from the fixed point is called the *radius* of the circle and the fixed point itself is called the *center* of the circle. In the cartesian coordinate system a point is designated by a pair of real numbers, say (c1, c2). Thus, we need three parameters to completely describe a circle: c1, the *x*-coordinate of the center; c2, the *y*-coordinate of the center; and r, the radius of the circle. Analytically, this tells us that the definition of a circle is the set of all points, (x, y) that satisfy the following equation based on the distance formula:

$$\#1: \quad (x - c1)^2 + (y - c2)^2 = r^2$$

Derive allows you to plot expressions such as #1 provided the values of the parameters are defined. For example, if you wished to see the graph of a circle with center at (-1, 2) and radius 3, you simply highlight expression #1, click on the SUB toolbar button and type in the respective new values for c1, c2, and r variables (parameters). Click on OK to replace the parameters with your values. Switch to the plot window and click on the plot toolbar button or click Insert > Plot. If you receive a message that says "Unable to Plot Highlighted Expression", open a 2D-plot window and click on Window > Approximate Before Plotting and try again. The resulting plot is the set of points, (x, y), satisfying the new values substituted in expression #1.

What if you know some points on a circle and want to find the equation of the circle? This is the reverse of our previous example. We need to find the values of the parameters c1, c2, and r. Once again, expression #1 is our road to success. The first question we need to address is, how many points uniquely determine a circle? Since we have three parameters to determine, it is sensible to guess that three points should do the job. Of course, the points can not be co-linear. Let's give it a try.

Suppose we wish to find a circle passing through the three points (1, 1), (-2, 0), and (-1. -1). First we copy expression #1 into the Derive Command Entry Line by highlighting it and pressing the F3 key. Substitute 1 for x and 1 for y. Press the Enter key. The expression will remain in the Command Entry Line. Alter the expression to reflect entering -2 for x and 0 for y. Press enter again. Repeat the process for the point (-1, -1). This results in the following three

equations.

#2: $(1 - c_1)^2 + (1 - c_2)^2 = r^2$

#3: $(-2 - c_1)^2 + (0 - c_2)^2 = r^2$

#4: $(-1 - c_1)^2 + (-1 - c_2)^2 = r^2$

Derive has a very powerful solver for systems of polynomial equations. Press Solve > System.
An inquiry box will ask you how many equations you wish to solve. Indicate 3. A display
similar to the one shown below appears. Since you have the equations as part of your Derive
worksheet, you need only enter the numbers of the expressions that have the three equations as
shown below.   Click on the blank space in the Solution Variables area and then click Solve.
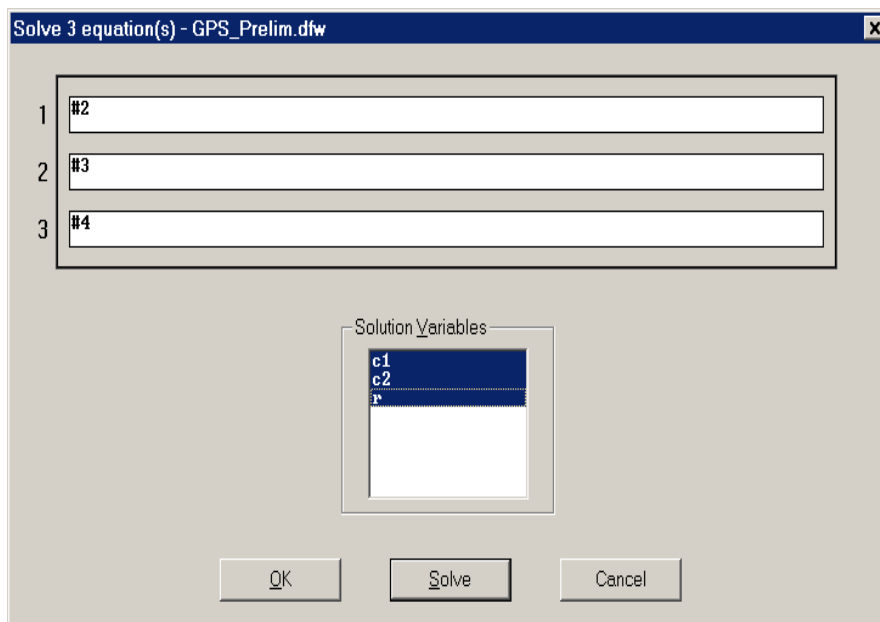
The following two expressions are displayed on your Derive worksheet. Note that in expression
#6 there are two answers. Why?

#5: $\text{SOLVE}\left[(1 - c_1)^2 + (1 - c_2)^2 = r^2, (-2 - c_1)^2 + (0 - c_2)^2 = r^2, (-1 - c_1)^2 + (-1 - c_2)^2 = r^2 \right], [c_1, c_2, r])$

#6: $\left[c_1 = -\dfrac{1}{2} \wedge c_2 = \dfrac{1}{2} \wedge r = \dfrac{\sqrt{10}}{2}, \quad c_1 = -\dfrac{1}{2} \wedge c_2 = \dfrac{1}{2} \wedge r = -\right.$

$$\left. \frac{\sqrt{10}}{2} \right]$$

At this point we enter the solution for c1, c2, and r into expression #1 and plot the circle.

#7:
$$\left(x - - \frac{1}{2} \right)^2 + \left(y - \frac{1}{2} \right)^2 = \left(\frac{\sqrt{10}}{2} \right)^2$$

Exercise
  1. Choose three arbitrary non colinear points on the plane and determine the equation of a circle through these three points. Graph the resulting circle.

Background - Step 2

Now that we have done a brief study of circles, we will consider how they may be applied to GPS System. Let's suppose that you are in a flat plane located at the point having grid coordinates (-1, 2), but you don't know that fact. Suppose there are transmitters located at grid coordinates, (3, 0), (-3, 1), and (0, 1). What you do know is that you are located at distances of the square root of 20, the square root of 5, and the square root of 2, respectively from these three points.   You will use that fact to locate your position on the plane.

The first thing that you do is write the equations for the three circles centered at the transmitters and having radii equal to their distance from your position on the plane.

#8: $(x - 3)^2 + (y - 0)^2 = \sqrt{20}^2$

#9: $(x - -3)^2 + (y - 1)^2 = \sqrt{5}^2$

#10: $(x - 0)^2 + (y - 1)^2 = \sqrt{2}^2$

Once we have the equations, it is natural to look at their graphs and find the intersection point(s). Highlight each of the expressions in the algebra window and plot them.   Note that even though there are only two independent variables, you need to plot all three of the expressions to find your position. Why?

You can solve the equations exactly using the Derive Solve > System command as we did before. The result of this command is shown below. Note that even though we had only two unknowns, we entered three equations.

#11: SOLVE$\left(\left[(x - 3)^2 + (y - 0)^2 = \sqrt{20}^2, (x - -3)^2 + (y - 1)^2 = \sqrt{5}^2, (x - \right. \right.$

$\left. \left. 0)^2 + (y - 1)^2 = \sqrt{2}^2 \right], [x, y] \right)$

#12:   [x = −1 ∧ y = 2]

At this point, let's review what you know about circles. In the following exercises draw a graph to illustrate that your answer is correct.

Exercises:
    2. Is it possible to find c1, c2, and r so that expression #1 has exactly one point, (x, y) that will satisfy the equation.
    3. Is it possible for two distinct circles to intersect at no point? one point?
two points?
    4. Is it possible for three distinct circles to intersect at no point? one point? two points?
   5. Why is it not possible to find three or more distinct circles that intersect in exactly three points?

A "2 Dimensional" GPS

NOTE: The velocity given for the signal sent from the transmitters has no relation to the actual speed with which signals are sent from GPS sattelites or radio transmitters. We are using this velocity only to keep the magnitude of the numbers we are dealing with more manageable so that the numbers do not get in the way of the exposition. In part II we will deal with a more realistic situation.

Prior to beginning this unit, resize your graphics screen so that the range on the x-axis is (-300, 900) with 4 intervals, and the range on the y-axis is (-200 , 800) with 5 intervals. Also,because accuracy is important in GPS calculations set the decimal accuracy to 10 digits. Use the Algebra Window's Declare > Simplification Settings to set the Precision Digits to 10.

You know that you are located at a point somewhere in this plane region. Assume that scale on the graphics screen is in miles. You activate your GPS to locate your position. It receives signals from three transmitters that have been placed in the plane at locations of loc1=(75, 150), loc2=(510, 110), and loc3=(150,20). The transmitters all know their own position and the exact time as determined by an atomic clock in the transmitter. The transmitter clocks are synchronized on a regular basis so that the time at each transmitter is absolutely accurate. Each transmitter broadcasts its coordinates and the time. The broadcast signal travels at 200 miles per second.

Your GPS is able to receive the signals and record the time that each signal is received. However, your GPS does not have a clock that is synchronized with the transmitters. Thus, there is a time time discrepency, say d, between the time recorded by your GPS receiver and the the actual time as determined by the synchronized atomic clocks in the transmitters. The GPS records the signals that it received. The times for each transmitter are subtracted from the time your GPS received them. These are recorded as t1, t2, and t3. The numbers correspond to the times from the respective transmitter locations. .

#13:   t1 := −0.8139056099

#14:   t2 := 0.7980845608

#15:   t3 := −0.4685532484

The actual times that it took to travel from the trasmitters to your GPS are t1+d, t2+d, and t3+d, where d is the time discrepency of your GPS. Thus, we have three unknown values: the x and y coordinates of your position and d, the time discrepency of your GPS.

Your distance from each of the transmitters is the velocity of the signal multiplied by the time that it took to travel to the GPS.

#16:   d1 := 200·(t1 + d)

#17:   d2 := 200·(t2 + d)

#18:   d3 := 200·(t3 + d)

But these distances are the radii of circles centered at each transmitter and passing through your position on the plane. We are now in the situation of Step 2 above. The one difference is that we now have three unknowns: x, y, and d. However we have three equations to solve. It may work out.

#19: $(x − 75)^2 + (y − 150)^2 = d1^2$

#20: $(x − 510)^2 + (y − 110)^2 = d2^2$

#21: $(x − 150)^2 + (y − 20)^2 = d3^2$

We simply use the Solve > System option as shown above and then approximate the result to obtain the following result. NOTE: the first result is not shown here. It is the exact symbollic solution to the system. It is rather messy since it involves expressions with sums of radicals and fractions of VERY large numbers. We want numbers that we can recognize. However, this will be a decimal approximation and we may loose some accuracy.

#22: $\text{SOLVE}([(x − 75)^2 + (y − 150)^2 = d1^2, (x − 510)^2 + (y − 110)^2 = d2^2,$

$(x − 150)^2 + (y − 20)^2 = d3^2], [x, y, d])$

#23:   [x = 452.5761121 ∧ y = 96.9671036 ∧ d = −1.092505996, x = 132 ∧ y =

145 ∧ d = 1.1]

Because we have three unknowns, we, once again, have two possible solutions to the system of equations. In this case we can rather easily eliminate one of them. Note that if d = -1.092505996 then each of the distances, d1, d2, and d3 will be negative. This is nonsense. We would have received the signals before they were sent. Thus, you can conclude that your position is at (132, 145) and the time discrepency is 1.1 second. Define d := 1.1 and graph expressions #19, #20, and #21. NOTE: This can be done by holding down the <Ctrl> key and

highlighting all three expressions followed by switching to the 2D-plot window and clicking on the plot button.

#24:   define d:=1.1 by editing this line
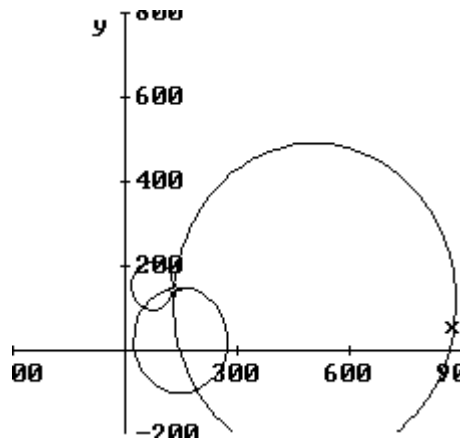
After viewing the graph clear the value for d by typing d := . This action frees the symbol d so that it can be a variable.

#25:   clear d by editing this line to read d :=

It is time to try one on your own. First clear the graphics screen by choosing the Edit > Delete All Plots from the 2D-plot menu. Assume that you are working with a different GPS receiver, so you have no idea of the time difference between the transmitter clocks and your GPS clock. You are still in the same region of the plane as in the example.   Here we go!

Exercise
    6. Your GPS receives signals from three transmitters located at positions in the plane given as (320, 190), (50, 100), and (570, 50), respectively. The time differences recorded by your GPS receivers and the three transmitters are given as t1 := 3.113941029 seconds, t2 := 3.113941029 seconds, and t3 := 4.102775637 seconds, respectively. Enter these times by editing the lines below.

#26:   enter t1 by editing this line

#27:   enter t2 by editing this line

#28:   enter t3 by editing this line

Edit the next three lines to give the formulas for the distance from each of the three transmitters. Don't forget that you need to include the time discrepency between the transmitters and your GPS in these formulas.

#29:   enter the distance to transmitter #1 by editing this line

#30:   enter the distance to transmitter #2 by editing this line

#31:   enter the distance to transmitter #3 by editing this line

Now that you have the distances, you can enter the three equations that you will need to solve in order to determine your position.on the plane. Edit the three expressions below so that they contain the equations.

#32:   equation 1

#33:   equation 2

#34:   equation 3

After editing the above lines, solve the equations you entered using the Solve>Equations option. In this case, the distances for either of the answers are positive. However, we can eliminate one of the aswers rather easily. It lies outside of the region of the plane where we know we are located. Write the value for d, the time discrepancy, and plot the three circles.

#35:   value for d here

As a final exercise, you will demonstrate your understanding of the process by constructing a problem on your own for others to solve.

Exercise
   7. Choose a set of 4 points that lie within the 2-D plot region of the plane that you have laid out and write down their coordinates. One of these points will be your position on the plane. The other three will be the locations of the three transmitters. Determine the distance of your point from each of the transmitters. Finally decide on a value for the variable, d, which is the time discrepency between the transmitters clocks and the clock on your GPS.
        a. Determine the time that it takes for each of the signals sent from the
           transmitters to reach your GPS.
        b. Using d, determine the values for t1, t2, and t3, the differences
           between the time sent by each transmitter and the time the signal
           is received.
        c. Write out your problem in a manner similar to the example and
           problem given above.
        d. Solve the problem and compare the answer with the coordinates
           for your position. This is a check that you correctly constructed
           the problem.

Carl Leinbach: leinbach@gettysburg.edu

#36:   InputMode := Word