

TI-89 Riemann Sum Activities for Calculus

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Teacher Notes

This package consists of four activities:

1. An activity introducing the concept of a Riemann sum by estimating distance traveled given a table of times and speeds
2. A TI-89 graphing calculator demonstration that uses Riemann sums to estimate the area under a curve
3. Setting up and finding limits of Riemann sums using a TI-89 graphing calculator
4. Converting Riemann sum limits to definite integrals and vice versa, verifying with a TI-89 graphing calculator

Ideally, the first two will be completed during one period. The first activity, estimating the distance traveled, should be done by students in small groups. Allow approximately fifteen minutes for completion of the task and another ten minutes for discussion. Hopefully, students will conclude that more readings would make the answer more reliable. To emphasize this point and introduce the more formal idea of Riemann sums to estimate the product of $f(x)$ and x where $f(x)$ is a rate of change, do the calculator demonstration outlined in the second activity. This should take approximately twenty minutes with discussion.

In the third activity, students will set up Riemann sums using right-hand endpoints and use a TI-89 graphing calculator to find the limits of the Riemann sums. Guide the students through Part I, showing them how to do problems 1 – 10 and pausing while they complete 11 – 16. Then have them do Part II in pairs.

Between the third and fourth activities, you will teach the lesson in your textbook that defines a definite integral as a limit of a Riemann sum.

The fourth activity provides an opportunity for the students to practice writing Riemann sums as definite integrals and vice versa. Once again, the students will pair up and use a TI-89 graphing calculator to check that they can translate between the two forms.

Activity 1. An Introduction to Riemann Sums¹

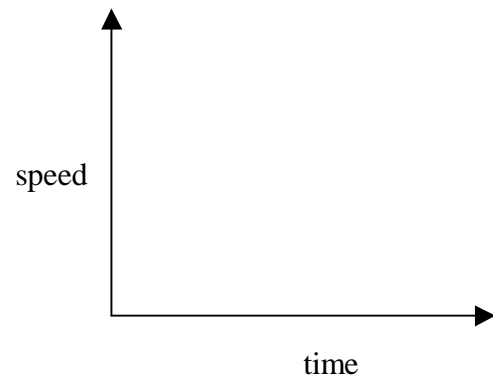
Leonard and Isaac are driving across the United States in their 1972 VW bus. Their odometer is broken and since they are short on funds they decide not to get it fixed. Their speedometer still works, so while Isaac is at the wheel Leonard records their speed periodically. He makes the observations recorded in the table at right.

<u>Time</u>	<u>Speed</u>
10:00	56 mph
10:30	63 mph
10:50	72 mph
11:30	40 mph
12:00	55 mph

1. How far did they drive between 10:00 and 12:00? (Be sure to explain how you arrived at your answer.)

2. How much confidence do you have in your answer to question 1?

3. Plot the data on the axes at the right. What part of the graph represents the distance traveled? Explain.



4. What would you need to know to improve your estimate of distance traveled?

¹ This problem is taken from the New Ideas in Calculus Program from the North Carolina School for Science and Math.

Activity 2. A TI-89 Graphing Calculator Demonstration of Riemann Sums

Load program area onto the TI-89 overhead graphing calculator and enter a function in y1. A good sample function is $y = \sqrt{x}$ with the following window settings:

xmin = -1 xmax = 4 xscl = 1
ymin = -1 ymax = 2 yscl = 1

Tell the students that the goal is to estimate the area under the curve above the x-axis from 1 to 3. In order to do this, the region is sliced vertically, forming rectangles whose areas can be calculated and added together. Begin by drawing 4 rectangles whose left endpoints are on the curve.

Run the program by going to the HOME screen and typing area() Then press <ENTER>

Reply to the questions

a?

1<ENTER>

b?

4<ENTER>

no. of subint

4<ENTER>

The picture shows four inscribed rectangles. Ask the students if the sum of the areas of these four rectangles over- or under-estimates the actual area.

Press <ENTER> to display the sum of the areas of the four rectangles. Have a student record the number of rectangles and the area on the board.

As you progress through the program demonstration, continue to press<ENTER>, recording results. After the midpoint sum is displayed, the program is finished so press <HOME>

Ask the students how to obtain a better approximation of the actual area.

Rerun the program with ten subdivisions.

Since the left sums underestimate the area and the right sums overestimate the area, it seems natural to average those two results together (*the Trapezoid Rule*). Compare this average with the result obtained from the midpoint sums and the actual area given by the calculator $\int_1^3 \sqrt{x} dx$

Note that the actual area is between the midpoint sums and the average of the left and right sums and that the midpoint sums is roughly twice as accurate. The weighted average $\frac{2}{3} \text{Mid} + \frac{1}{3} \text{Trap}$ is quite close to the actual area. By the way this weighted average is actually *Simpson's Rule*.

It is a good idea to repeat this demonstration on a second curve, one whose concavity is reversed so that students do not always believe that the left endpoint method underestimates while the right endpoint overestimates the area. To enter a new function, you must quit the program, enter the new function, and run the program again. A good function to use with the same window settings, left and right endpoints and partitions is $y = 6e^{-x}$.

Activity 3.

Name _____ Pd _____

Using a TI-89 to Find Limits of Riemann Sums

In this activity you will find the area of a plane region bounded by a non-negative function and the x-axis on a closed interval. After sketching the region, you will partition the interval into n subintervals, and then write an expression that sums up the areas of the rectangles formed by using the right-hand endpoints.

Part I.

- Sketch the region bounded by $f(x) = 2x$ and the x-axis over $[1, 2]$.

Now find an expression for the area of the i -th rectangle, using right-hand endpoints.

- If the interval $[1, 2]$ is partitioned into n equal subintervals, what is the width of one subinterval? $\Delta x =$ _____
- Find an expression for the right-hand endpoint of the i -th subinterval. $c_i =$ _____
- Find an expression for $f(c_i)$. $f(c_i) =$ _____
- Find an expression for the area of the i -th rectangle. $f(c_i)\Delta x =$ _____

To find a Riemann sum for f on $[1, 2]$, use your TI-89 graphing calculator.

- To build the sum of the n such rectangles as i varies from 1 to n , go to the catalog and press Σ (you are already in alpha mode). Page down three times and down arrow to $\Sigma(\text{sum})$. Selecting the summation through the catalog shows, at the bottom of the screen, the four arguments expected: $\text{EXP}, \text{VAR}, \text{LOW}, \text{HIGH}$
- Press $\langle \text{ENTER} \rangle$ and fill in the expression for the area of the i -th rectangle found in #5 and complete the sum letting i vary from 1 to n . $\Sigma(2(1+i/n)(1/n), i, 1, n)$
- Estimate the area when n is 5 by moving to the right of entry line and typing $\mid_{n=5}$
Repeat for $n = 10$ and $n = 100$
at $n = 5$ Riemann sum is _____ at $n = 10$ Riemann sum is _____ at $n = 100$ Riemann sum is _____
- Now evaluate the limit of the Riemann sum as n increases without bound.
 $\text{limit}(\Sigma(2(1+i/n)(1/n), i, 1, n), n, \infty)$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(2 + \frac{i}{n} \right) \frac{1}{n}$$

- Finally, what is the area of the region sketched in #1? _____

Suppose $f(x) = 2x$ is instead partitioned into n subintervals over $[3, 7]$. Find each of the following:

- 11. the width of one subinterval $\Delta x =$ _____
- 12. an expression for the right-hand endpoint of the i -th subinterval? $c_i =$ _____
- 13. an expression for $f(c_i)$ $f(c_i) =$ _____
- 14. an expression for the area of the i -th rectangle $f(c_i)\Delta x =$ _____
- 15. the Riemann sum when $n = 5$ _____
- 16. Using the limit process, find the area of the region over $[3, 7]$ _____

Part II. Use the process outlined in Part I to find the area of each region bounded by the given function and the x -axis on $[a, b]$.

Function, $[a, b]$	Sketch	Limit expression of the Riemann sum	Area
1. $f(x) = 3x$ on $[0, 2]$		_____	_____
2. $f(x) = x^2$ on $[0, 3]$		_____	_____
3. $f(x) = 2x^2$ on $[1, 3]$		_____	_____
4. $f(x) = 1 - x^2$ on $[-1, 1]$		_____	_____
5. $f(x) = 2x - x^3$ on $[0, 1]$		_____	_____

Activity 4.

Name _____ Pd _____

Translating between Definite Integrals and Riemann Sums

Follow the TI-89 calculator instructions below in order to evaluate $\int_1^2 (x+1)dx$.

Select F3 (calculus) | 2: \int (integrate) and press <ENTER>

Type in the four parameters and the closing right parenthesis $x+1,x,1,2$ <ENTER>

Note that holding down the green diamond key while pressing <ENTER> gives a decimal approximation.

What is the value of this definite integral? _____

What is the corresponding Riemann sum for this definite integral?

Since the width is $\Delta x = \frac{1}{n}$ and $\bar{x}_i = 1 + \frac{i}{n}$, the height is $f(\bar{x}_i) = [(1 + \frac{i}{n}) + 1] = 2 + \frac{i}{n}$.

So the corresponding Riemann sum is $\lim_{n \rightarrow \infty} \sum_{i=1}^n (2 + \frac{i}{n})(\frac{1}{n})$.

To see if the Riemann sum is correct, evaluating the limit you built. What is the result? _____

Part A. In each of the following problems, translate the definite integral into a Riemann sum. Check that both forms give the same evaluation. Write down the evaluation.

	Limit expression for Riemann sum	Evaluation
1. $\int_1^3 (2x+1)dx$	_____	_____
2. $\int_2^4 (3x^2 - 1)dx$	_____	_____
3. $\int_{-2}^0 (x+3)^2 dx$	_____	_____
4. $\int_0^\pi \sin x dx$	_____	_____

Part B. In each of the following problems, translate the Riemann sum into a definite integral. Check that both forms give the same evaluation. Write down the evaluation.

	Definite Integral	Evaluation
5. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[3\left(\frac{3i}{n} + 2\right) - 8 \right] \left(\frac{3}{n}\right)$	_____	_____
6. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[5\left(\frac{5i}{n} - 2\right)^3 - 1 \right] \left(\frac{5}{n}\right)$	_____	_____
7. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[3\left(\frac{2i}{n} - 2\right) + 4 \right] \left(\frac{2}{n}\right)$	_____	_____
8. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\cos\left(\frac{i\pi}{6n}\right) \right] \left(\frac{\pi}{6n}\right)$	_____	_____

A Riemann sum in simplified form should be unsimplified before being translated into a definite

integral. For example, $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i^3}{n^4} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right) = \int_0^1 x^3 dx$

9. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\sqrt{i}}{\sqrt{n^3}}$	_____	_____
10. $\lim_{n \rightarrow \infty} \left[\frac{1}{n^3} + \frac{4}{n^3} + \frac{9}{n^3} + \dots + \frac{n^2}{n^3} \right]$	_____	_____

Solutions

Introduction to Riemann Sums Worksheet: Answers will vary but they should be in the neighborhood of 113.3 miles. The area under the curve represents distance because rate x time = distance.

Limits of Riemann Sums Worksheet: Part I: 2. $\Delta x = \frac{1}{n}$ 3. $c_i = 1 + \frac{i}{n}$

4. $f(c_i) = 2 + \frac{2i}{n}$ 5. $\frac{1}{n} \left(2 + \frac{2i}{n} \right)$ 8. 16/5, 31/10, 301/100

9 & 10. 3 11. $\Delta x = \frac{4}{n}$ 12. $c_i = 3 + \frac{4i}{n}$

13. $f(c_i) = 6 + \frac{3i}{n}$ 14. $\frac{4}{n} \left(6 + \frac{3i}{n} \right)$ 15. $\sum_{i=1}^n \left(\frac{4}{n} \right) \left(6 + \frac{3i}{n} \right)$

16. 40

Part II: 1. $\sum_{i=1}^n \left(\frac{2}{n} \right) \left(\frac{6i}{n} \right)$, 6 2. $\sum_{i=1}^n \left(\frac{3}{n} \right) \left(\frac{9i^2}{n^2} \right)$, 9

3. $\sum_{i=1}^n \left(\frac{4}{n} \right) \left(1 + \frac{2i}{n} \right)^2$, $\frac{52}{3}$ 4. $\sum_{i=1}^n \left(\frac{2}{n} \right) \left(1 - \left(-1 + \frac{2i}{n} \right)^2 \right)$, $\frac{4}{3}$

5. $\sum_{i=1}^n \left(\frac{1}{n} \right) \left(2 \left(\frac{i}{n} \right) - \frac{i^3}{n^3} \right)$, $\frac{3}{4}$

Translating Between Definite Integrals and Riemann Sums: 1. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2}{n} \right) \left(2 \left(1 + \frac{2i}{n} \right) + 1 \right)$, 10

2. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2}{n} \right) \left(3 \left(2 + \frac{2i}{n} \right)^2 + 11 \right)$, 54 3. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2}{n} \right) \left(-2 + \frac{2i}{n} + 3 \right)^2$, $\frac{26}{3}$

4. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \left(\frac{\mathbf{p}}{n} \right) \left(\frac{\mathbf{p}}{n} \right)$, 2 5. $\int_2^5 (3x - 8) dx$, $\frac{15}{2}$

6. $\int_{-2}^3 (5x^3 - 1) dx$, $\frac{305}{4}$ 7. $\int_{-2}^0 (3x + 4) dx$, 2

8. $\int_0^{\frac{\mathbf{p}}{6}} \cos x dx$, $\frac{1}{2}$ 9. $\int_0^1 \sqrt{x} dx$, $\frac{2}{3}$

10. $\int_0^1 x^2 dx$, $\frac{1}{3}$