Circles- Angles & Arcs

Math Objectives

- Students will define and identify central angles, major and minor arcs, intercepted arcs, and inscribed angles of a circle.
- Students will determine and apply the following relationships:
- Two inscribed angles intercepting the same arc have the same measure.
- An inscribed angle measure of 90° results in the endpoints of the intercepted arc lying on a diameter.
- The measure of an angle inscribed in a circle is half the measure of the central angle that intercepts the same arc.
- Students will try to make a connection with how to understand these topics in IB Mathematics courses on their final assessments

Vocabulary

- central angle
- major, minor, and intercepted arc
- inscribed angle

About the Lesson

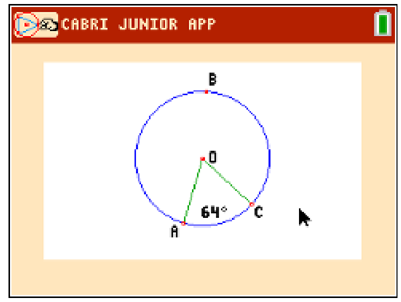
- This lesson is aligning with the curriculum of IB Mathematics Applications and Interpretations SL/HL and IB Mathematics Approaches and Analysis SL/HL
- This falls under the IB Mathematics Core Content Topic 3 Geometry and Trigonometry:
  - **3.4** The circle, length of an arc, area of a sector (*The AA course also includes radian measure in this section as well)
- As a result, students will apply this to real world situations.

Teacher Preparation and Notes

- This activity is done with the use of the TI-84 family as an aid to the problems, specifically using Cabri Jr, familiarity will help

Activity Materials

- Compatible TI Technologies: TI-84 Plus*, TI-84 Plus Silver Edition*, TI-84 Plus C Silver Edition, TI-84 Plus CE



Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

Student Activity

Circles_Angles_and_Arcs_Student-84.pdf

Circles_Angles_and_Arcs_Student-84.doc

ANGARC1 (Cabri Jr. App)

ANGARC2 (Cabri Jr. App)

ANGARC3 (Cabri Jr. App)

** with the latest operating system (2.55MP) featuring MathPrint™ functionality.*

Discussion Points and Possible Answers

Problem 1 – Central Angles & Inscribed Angles

Start the Cabri Jr. application. Then, open the file ANGARC1.

1. Drag point *A* or point *C*. Describe the changes that occur in the figure as you drag the point.

   Answer: The angle measurement changes but is never more than 180°. The solid part of the circle is always in the interior of the angle.

2. Angle *AOC* is called a central angle. Explain why you think this is so.

   Answer: The vertex is at the center of the circle.

Note: The measure of a central angle is equal to the measure of the intercepted arc. Since there is no "arc measure" tool, the central angle measure is used as the arc measure.

An angle intercepts an arc of a circle if each endpoint of the arc is on a different ray of the angle and the other points of the arc are in the interior of the angle.

As you move point *A* or point *C*, the central angle $\angle AOC$ intercepts a minor arc *AC*. The measure of the minor arc equals the measure of the central angle. The larger remaining arc, *ABC*, is called a major arc.

3. a. Move point *A* or point *C* to help you complete the table.

   Sample answer: The completed table is below. The final row of students' tables will vary.

$\angle$***AOC***	**arc *AC***	**arc *ABC***	**arc *AC* + arc *ABC***
50°	50°	**310°**	**360°**
100°	100°	**260°**	**360°**
110°	**110°**	250°	**360°**
(Choose an angle.) **60°**	**60°**	**300°**	**360°**

b. With a classmate, discuss what is true about the measure of arc *AC* + arc *ABC*, the sum of the measures of the minor and major arcs. Share your results with the class.

Answer: The measure of arc *AC* + arc *ABC* always equals 360°.

4. In a circle, the measure of a central angle $\angle AOC$ is *n*°.

   a. Find the measure of the minor arc that is intercepted by the central angle. Explain to a classmate how you know this to be true.

   Answer: The minor arc measures *n*° because an intercepted arc has the same measurement as its central angle.

   b. Find the measure of the major arc. Explain to a classmate how you know this to be true.

   Answer: The major arc measures (360 – *n*)° because the sum of the measures of the major and minor arcs is 360°.

5. Now that you have found the measure of the central angle and the degree measures of both minor and major intercepted arcs, discuss with a classmate how you may be able to find the "length" of those arcs. Discuss the information you would need to find these lengths and the process this would entail.

   Answer: One method to be discussed is setting up a proportion between the measure of the central angle and the full rotation of the circle (360°), and the length of the intercepted arc and the circumference of the circle. Depending on the level of student, you can also discuss the use of the arc length formula $l = \theta r$, as long as θ is in radians. For both of these methods, the radius of the circle would have to be known.

Once again from the Cabri Jr. app, open the file ANGARC2.

6. Angle *ABC* is called an inscribed angle because $\overline{BA}$ and $\overline{BC}$ are chords of the circle and vertex *B* is on the circle. Move your cursor to point B and press alpha. Drag point *B* around the circle.

   a. As point *B* is moved around the circle, discuss with a classmate what you notice about the measure of $\angle ABC$?

   Answer: Angle *ABC* has the same measure until it intercepts the other arc. While point *B* is moved around on that arc, $\angle ABC$ will remain the same.

Teacher Tip: Some students may recognize that the two angles' measures sum to 180°.

b. With a classmate, discuss why the $m\angle ABC$ changes when point *B* is moved from one arc to the other. Share your reasoning with the class.

Answer: The angle measure changes because the intercepted arc changes. The intercepted arc is always in the interior of the inscribed angle.

c. Move point *A* or point *C* until $\angle ABC$ is a right angle. Discuss with a classmate what is special about the intercepted arc ADC and $\overline{AC}$.

Answer: The arc measure is 180° and the arc is a semicircle. *AC* is a diameter.

Teacher Tip: When students reach the right angle, the diameter should show up as a dotted segment.

Once again from the Cabri Jr. app, open the file ANGARC3.

Angle *ABC* intercepts arc *AC*. Drag point *D* to various locations outside the circle, on the circle, inside the circle, and at the center *O*.

7. Place point *D* on the circle so that $\angle ADC$ intercepts the same arc as $\angle ABC$.

a. Discuss with a classmate what you notice about the measures of $\angle ABC$ and $\angle ADC$.

Answer: The angle measures are the same. The angles are congruent. The ratio of the angle measurements is 1.

Teacher Tip: Make sure both angles intercept the same arc. Some students may incorrectly place point *D* on the (bold) arc intercepted by $\angle ABC$. Point *D* should "snap" to the circle when it gets close.

b. Discuss with a classmate what happens to the angles if you move point *A* or point *C*.

Answer: The angle measures change. The angles are congruent and the ratio of the measure of $\angle ADC$ to the measure of $\angle ABC$ is 1 if the angles intercept the same arc. The angles are not congruent and the ratio is not 1 if the angles do not intercept the same arc.

Teacher Tip: Some students may note that the angle measurements are the same whenever *AD* and *BC* intersect or "criss-cross" and are not the same when they don't. This observation is important but needs to be related to intercepted arcs. Some students may notice that the angles are supplementary when they do not intercept the same arc. This property is addressed in problem 10. If m∠*ABC* equals 90°, then m∠*ADC* equals 90° whether or not they share the same intercepted arc.

8. Place point *D* at the center of the circle, making sure that ∠*ADC* intercepts the same arc as ∠*ABC*.

   a. Describe the relationship between the measures of inscribed ∠*ABC* and central ∠*ADC*.

   Answer: The measure of the central angle is double the measure of the inscribed angle. The measure of the inscribed angle is half the measure of the central angle. The ratio of the measure of ∠*ADC* to the measure of ∠*ABC* is 2.

   b. Discuss with a classmate what happens to the angles if you move point *A* or point *C*.

   Answer: The measure of the inscribed angle is half the measure of the central angle (the ratio of the measure of ∠*ADC* to the measure of ∠*ABC* is 2), as long as both angles intercept the same arc.

   c. Complete these conjectures:

- The measure of the inscribed angle is ________________ the measure of the central angle.
- The measure of the inscribed angle is ________________ the measure of the intercepted arc.

Answer: The measure of the inscribed angle is **half** the measure of the central angle and **half** the measure of the intercepted arc.

Teacher Tip: For a proof of the Inscribed Angle Theorem: In the simplest case, one leg of the inscribed angle is a diameter of the circle so it passes through the center of the circle. Since that leg is a straight line, the supplement of the central angle equals $180° - 2\theta$. Drawing a segment from the center of the circle to the other point of intersection of the inscribed angle produces an isosceles triangle, made from two radii of the circle and the second leg of the inscribed angle. Since two angles in an isosceles triangle are equal and since the angles in a triangle sum to 180°, it follows that the inscribed angle equals θ, half of the central angle.

9. Leona said, “Since a central angle can never measure more than 180°, I know an inscribed angle can never measure more than 90°.” Discuss with a classmate if you agree or disagree. Explain why.

   Answer: I disagree because the central angle always intercepts a minor arc, but an inscribed angle can intercept a major arc.

10. Place point *D* on the circle so that *ABCD* is a quadrilateral.

    a. Discuss with a classmate what you notice about the sum of the measures of $\angle ABC$ and $\angle ADC$. Share your results with the class.

       Answer: The sum of the measures of $\angle ABC$ and $\angle ADC$ is 180°.

    b. Discuss with a classmate what you notice about the sum of the measures of the angles if you move point *A* or point C.

       Answer: As long as *ABCD* remains a quadrilateral, the sum of the measures of $\angle ABC$ and $\angle ADC$ remains 180°.

    c. Discuss with a classmate what you notice about arcs *ABC* and *ADC*.

       Answer: One is a major arc and one is a minor arc. Together the arcs make a circle. The measures of the arcs sum to 360°.

    d. Describe how the relationship between arcs *ABC* and *ADC* explain the sum of the measures of inscribed $\angle ABC$ and $\angle ADC$.

       Answer: The sum of the measures of the major and minor arcs is 360°. Since the measures of the inscribed angles are half the measures of their intercepted arcs, the angles are supplementary.

> **Teacher Tip:** A quadrilateral inscribed in a circle has the special name “cyclic quadrilateral.”

Further IB Extension

A surveyor standing on the bank of the Reedy River measures the equal distance to the far left end and the far right end of the Liberty Bridge in Greenville, South Carolina, at a central angle of 120°. She found this distance to be 160 ft. See the diagram below (not to scale).

A sector is formed with these two equal distances and the bridge.

(a) Find the arc length of the inner guardrail and the arc length of the outer guardrail.

Answer: Method 1 $\frac{120°}{360°} = \frac{arc}{2\pi(165)}$ arc (inner guardrail) $\approx 346\ ft.$

Method 2 $l = \theta r, \theta = 120 \cdot \frac{\pi}{180} = \frac{2\pi}{3},\ l = \left(\frac{2\pi}{3}\right) \cdot 165\ \approx 346\ ft.$

(b) Find the arc length of the outer guardrail, given that the bridge is a uniform width of 12 ft. across.

Answer: Method 1 $\frac{120°}{360°} = \frac{arc}{2\pi(177)}$ arc (inner guardrail) $\approx 371\ ft.$

Method 2 $l = \theta r, \theta = 120 \cdot \frac{\pi}{180} = \frac{2\pi}{3},\ l = \left(\frac{2\pi}{3}\right) \cdot 177\ \approx 371\ ft.$

(c) Find the area of the walkable portion of the bridge.

Answer: Subtract the larger sector (with outer guardrail) and the smaller sector (with inner guardrail), using the formula: $A = \frac{1}{2}\theta r^2$

$\left(\frac{1}{2} \cdot \frac{2\pi}{3} \cdot 177^2\right) - \left(\frac{1}{2} \cdot \frac{2\pi}{3} \cdot 165^2\right)\ \approx 4297.699 \ldots \approx 4300\ ft.^2$

Wrap Up:

Upon completion of the discussion, the teacher should ensure that students understand:

- Two inscribed angles intercepting the same arc have the same measure.
- An inscribed angle measure of 90° results in the endpoints of the intercepted arc lying on a diameter.
- The measure of the inscribed angle is half the measure of the central angle that intercepts the same arc.

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