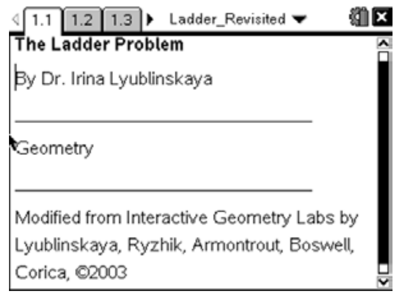
The Ladder Revisited.

by Dr. Irina Lyublinskaya, College of Staten Island, CUNY, NY

Activity overview



In this activity students explore the locus of mid-point of the hypotenuse of a fixed length geometrically and algebraically and discover that the median a right triangle is equal to half the length of the hypotenuse. Students then prove this property. Teacher is encouraged to have students work through the following stages of this problem.

- visualize the path in your mind without the aid of any materials
- describe or draw a diagram of what has been visualized; the diagram should <u>not</u> be a sketch of the situation
- investigate the problem with TI-Nspire technology; by this stage students should start to consider how the proof of their conjecture will be approached
- give a convincing argument for the solution to the problem

Statement of the problem: *A ladder leans upright against a wall that is perpendicular to the floor. A painter is standing at the midpoint of the ladder. If the bottom of the ladder begins to slide away from the wall, what path will his feet follow?*

Concepts

The median to the hypotenuse of a right triangle is equal to one half the measure of the hypotenuse

Teacher preparation

Before carrying out this activity teacher should review with the students the following concepts: right triangles, congruent triangles, median of a right triangle, circle, and equation of a circle. The screenshots on pages 3-7 demonstrate expected student results. Refer to the screenshots on pages 7-8 for a preview of the student TI-Nspire document (.tns file).

Classroom management tips

- *This activity is designed to be **teacher-led** with students following along on their handhelds. You may use the following pages to present the material to the class and encourage discussion. Note that the majority of the ideas and concepts are presented only in **this** document, so you should make sure to cover all the material necessary for students to comprehend the concepts.*
- *The ideas contained in the following pages are intended to provide a framework as to how the activity will progress. Suggestions are also provided to help ensure that the objectives for this activity are met.*
- *Students may answer the questions posed in the .tns file using the Notes application or on a separate sheet of paper.*
- *In some cases, these instructions are specific to those students using TI-Nspire handheld devices, but the activity can easily be done using TI-Nspire computer software.*

TI-Nspire Applications

Graphs, Geometry, Lists & Spreadsheets, Notes.

by: Dr. Irina Lyublinskaya

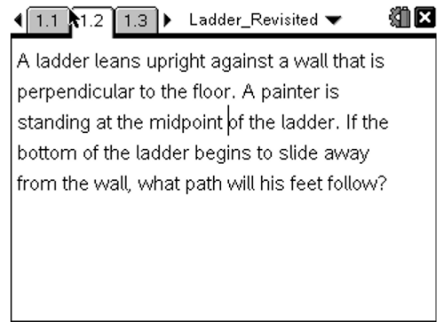
Grade level: secondary
Subject: mathematics
Time required: 45 to 90 minutes

Materials: TI-Nspire

Step-by-step directions

Problem 1 – Locus of the Midpoint of the Hypotenuse.

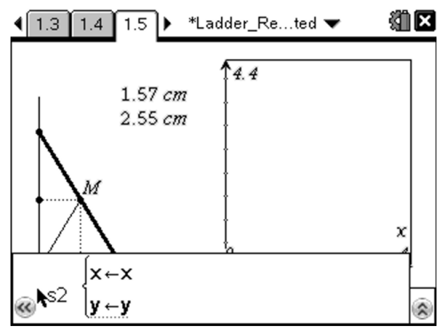
Step 1. Students open file Ladder_Revisited.tns, read problem statement on page 1.2, and move to page 1.3 where they need to answer first question.



Q1. Close your eyes. Can you visualize the trajectory? If so describe or draw your conjecture about this path (locus).

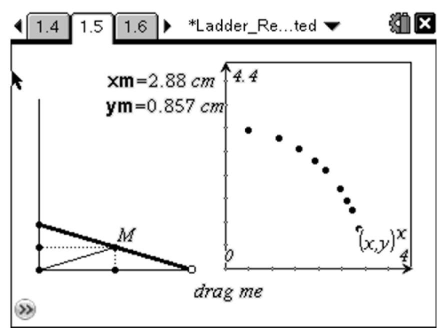
A. The correct answer is: the locus of point M is a quarter circle or a 90° arc of a circle.

At this initial stage of the problem many students may offer incorrect answers. They should not be corrected, since they can correct themselves after the exploration using TI-Nspire.



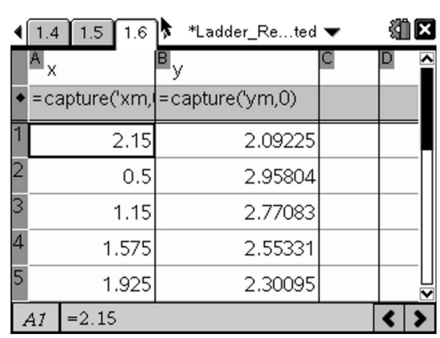
Step 2. Students should move to page 1.4 and read instructions for the exploration, then they move to page 1.5.

The ladder is modeled on page 1.5. Point M is the midpoint of the hypotenuse. Coordinates of the point M are measured and stored as (*xm*, *ym*). Students can drag the point on the horizontal segment that is indicated as "drag me". While they do that, they should observe the path that point M takes and how the coordinates of the point M are changing.



In order to capture coordinated and plot them on the scatter plot, the spreadsheet is set up on page 1.6 in a such way that coordinate *xm* can be manually captured to the 1st column as list variable *x*, and coordinate *ym* can be manually captured to the 2nd column as list variable *y*.

Step 3. Before capturing data, students show set up scatter plot by clicking ctrl G to show entry line. Then choose menu, 3: Graph Type, 4:Scatter Plot, and press Enter.



Step 4. Press enter and variables choice will be displayed. Choose *x* for x-list. Use tab and set up *y* – list. Then press enter and use ctrl G to hide entry line.

Step 5. Adjust the ladder and click ctrl . to capture the coordinates. The point will appear on the scatter plot. Repeat that for at least 6 more points.

Students should also check the lists on page 1.6 and compare data in the data table with the scatter plot.

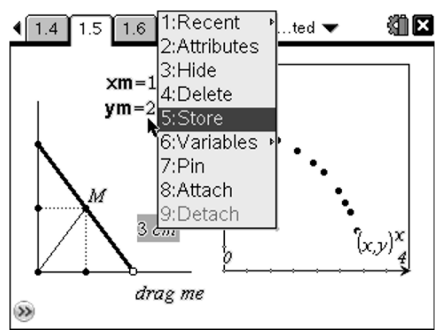
Q2. Based on collected data what is the shape of the locus of the point M?

A. It is expected that all students will be able to recognize that the locus is a quarter circle. The complete answer should include the statement about the center of this circle to be at the corner, and the radius equal to the median of the right triangle and/or to the half of the length of the hypotenuse.

Q3. Can you determine the function that describes this locus? Plot this function along with the scatter plot on page 1.8.

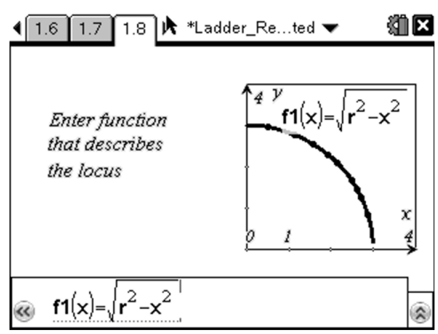
A. The equation is $y = \sqrt{r^2 - x^2}$, where *r* is the distance from the corner to point M.

It is recommended that students measure the length of the segment connecting point M and the vertex of the right triangle and store this measurement as *r*. Then they can enter function in terms of *r* instead of numerically. In order to do this, students should follow the steps below:



Step 6. Go back to page 1.5. Choose (menu), 7: Measurements, 1: Length. Click on the segment, move the cursor to the top of the screen and press (enter). The distance will be displaced.

Step 7. Click (esc) to quit measurement tool, click on the measurement once so it is highlighted, then click (ctrl)(menu) and choose 5: Store. In the open window type r for variable and press (enter).



Step 8. Go to page 1.8, enter for f1(x) $= \sqrt{r^2 - x^2}$, and press (enter). The graph of the function will be displayed along with the scatter plot.

Step 9. Move to page 1.9, read and answer question, then move to page 1.10 and provide proof or convincing argument for your conjecture.

Q4. Formulate your final conjecture for this problem and prove it.

A. The final conjecture that students formulate should include all the following statements:

- The locus is a 90° arc of a circle in 1st quadrant
- The radius of the circle is equal to the length of the median

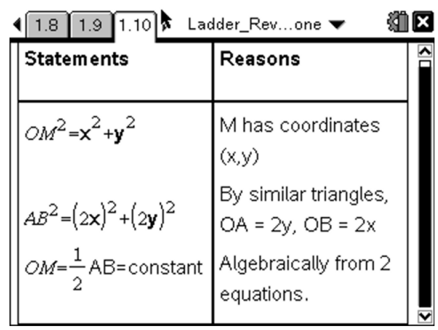
of the right triangle

- o The radius of the circle is equal to the half of the length of the hypotenuse.

The possible geometric proof for this statement is following:

Since M is the midpoint, AM = MB. Draw median OM. In ΔAOB, draw the midsegments DE and ME. We know that ME $\cong$ DO by the midsegment property and OE $\cong$ OE by reflexive property. We are given OA perpendicular to OB therefore ME is perpendicular to OB since if a transversal is perpendicular to one of two parallel lines, then it is perpendicular to the second. $\angle$DOE and $\angle$MEO are right angles and are congruent. ΔDOE $\cong$ ΔMEO by SAS and therefore by corresponding parts, DE $\cong$ OM. Since DE $\cong$ AM and DE $\cong$ MB by midsegments property, we conclude that OM $\cong$ AM and OM $\cong$ MB by transitivity of congruent segments. Now, since AB is a constant, OM is a constant. M is on the circle with center O and radius OM.

The possible algebraic proof for this problem is following:



Statements	Reasons
$OM^2=\mathbf{x}^2+\mathbf{y}^2$	M has coordinates (x,y)
$AB^2=(2\mathbf{x})^2+(2\mathbf{y})^2$	By similar triangles, OA = 2y, OB = 2x
$OM=\frac{1}{2}$AB=constant	Algebraically from 2 equations.

To show that OM is the radius of a circle, we need to show that OM is a constant. In the diagram, M has coordinates (x, y).
Know: $\mathrm{OM}^2 = x^2 + y^2$. Also, $\mathrm{AB}^2 = \mathrm{OA}^2 + \mathrm{OB}^2$. Then,

$$\mathrm{AB}^2 = (2\mathrm{y})^2 + (2\mathrm{x})^2$$
$$\mathrm{AB}^2 = 4(\mathrm{y}^2 + \mathrm{x}^2)$$
$$\frac{1}{4}\mathrm{AB}^2 = x^2 + y^2$$

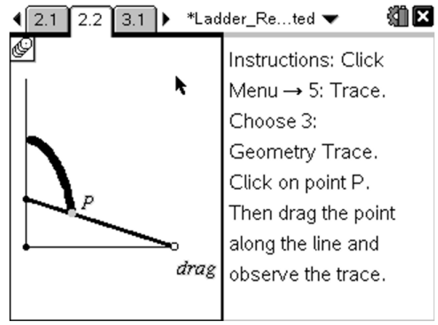
So, $\mathrm{OM}^2 = \frac{1}{4}\mathrm{AB}^2$ which implies than OM is a constant.

Students can use provided template in the TI-Nspire document to record their proofs, or they can record that on paper.

Activity extensions

Problem 2 – Extension 1.

Step 1. Students read the problem statement on page 2.1, make a conjecture and then move to page 2.2 to explore this situation.



Q5. Statement of the problem: What if the painter is standing at any other point on the ladder? What is the trajectory (locus) of this new point?

Step 2. Students should choose (menu), 5: Trace, 3: Geometry Trace. Press (enter) and click on point P that represents the painter's feet

Step 3. Move the cursor over the "drag me" point, use (ctrl)(click) and then drag the point along the horizontal and observe the trace left by the point P.

A. The locus is an arc of an ellipse with the center at the corner.

Step 4. Choose (menu), 5: Trace, 4: Erase Geometry Trace. Move point P in a different position, and repeat steps 2 and 3.

Comment: teacher may decide to leave out the proof of this extension. For the reference the proof is provided below:

Let MA = a, MB = b. Draw perpendiculars from point M to OA and to OB. MD $\perp$ OA, ME $\perp$ OB. Let $\angle$AMD = φ, then $\angle$ABO = φ. If coordinates of the point M are x and y, we have:

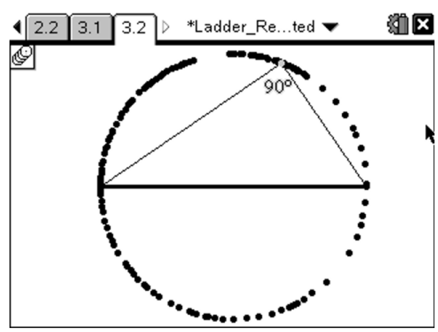
x = MD = $a \cos\varphi \Rightarrow \cos\varphi = x/a$

y = ME = $b \sin\varphi \Rightarrow \sin\varphi = y/b$

Since $\cos^2\varphi + \sin^2\varphi = 1$, then $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, so point M is on the arc of the ellipse with the center at O.

Problem 3 – Extension 2.

Step 1. Students read the problem statement on page 3.1, make a conjecture and then move to page 3.2 to explore this situation.



Q6. What is the locus of the right angle vertices of right triangles with same hypotenuse?

Step 2. Students should choose (menu), 5: Trace, 3: Geometry Trace. Press (enter) and click on the vertex of the right triangle.

Step 3. They could them use (ctrl)(click), drag this point and observe its trace.

A. The locus of the right angle vertices of right triangles with the same hypotenuse is the circle with the center in the mid-point of the hypotenuse and radius equal to the half of the hypotenuse

Texas Instruments

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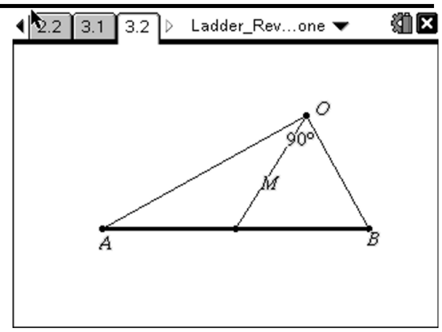
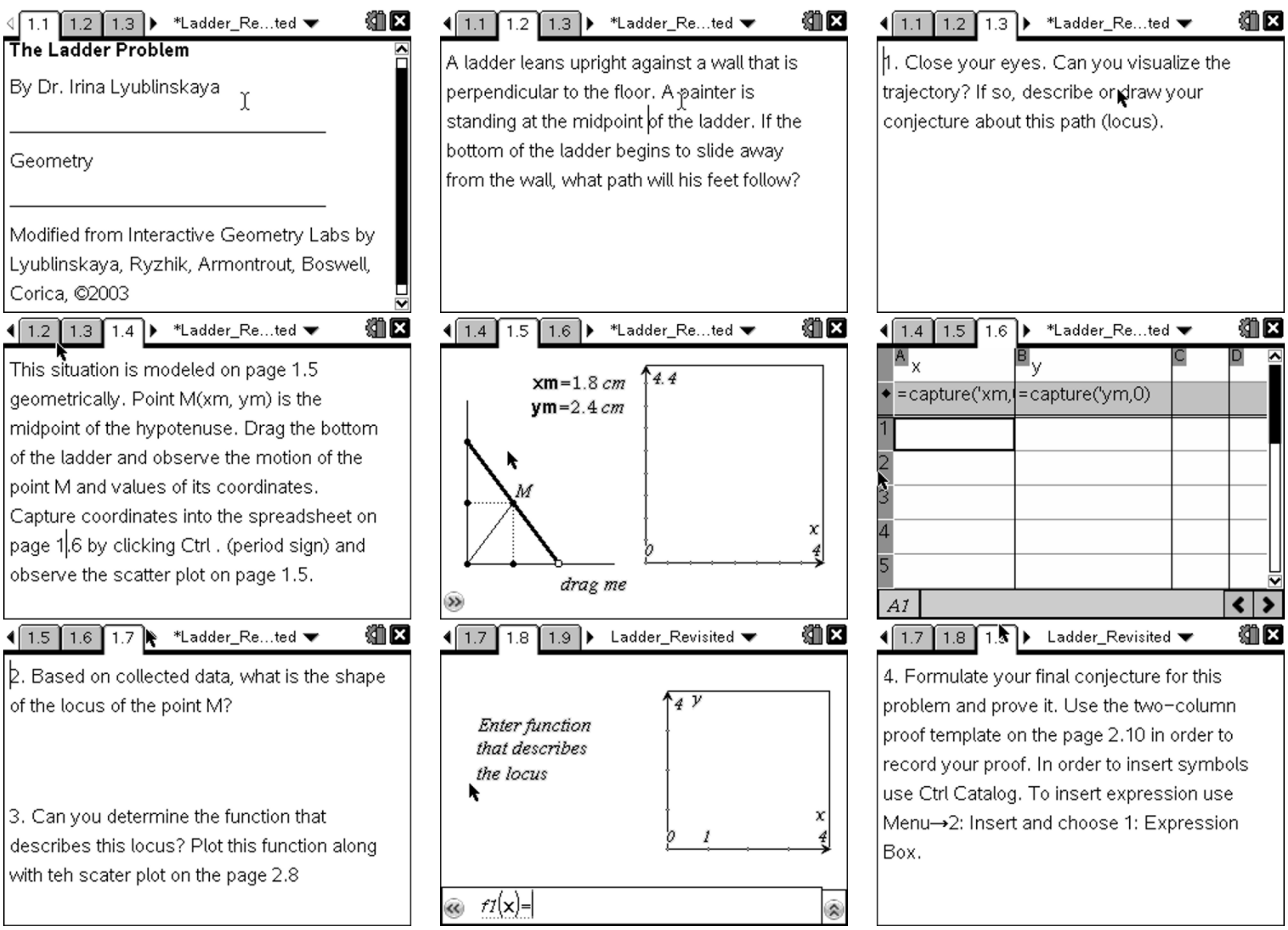
Materials: TI-Nspire

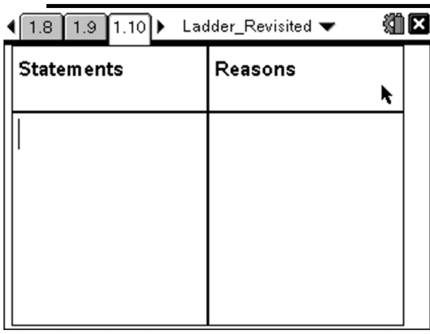
length.

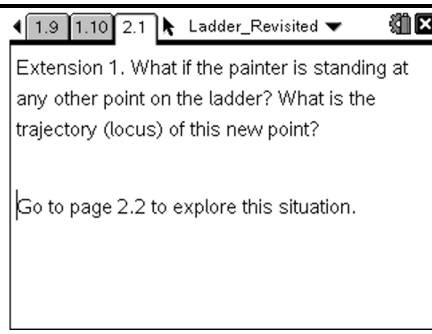
Proof: Construct a mid-point M on AB; MA = MB. Then, OM is a median of the right triangle, so OM = MA = MB = *r*. Then, for any right triangle with the hypotenuse AB, the vertex O will be at the distance *r*, which means that vertices of all right triangles form a circle with the center at M and radius *r*.

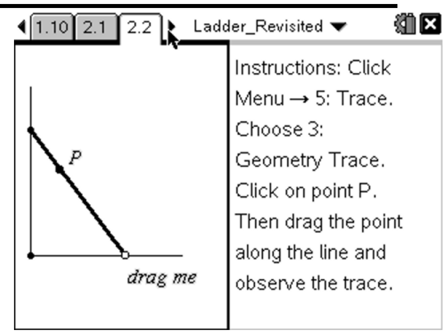


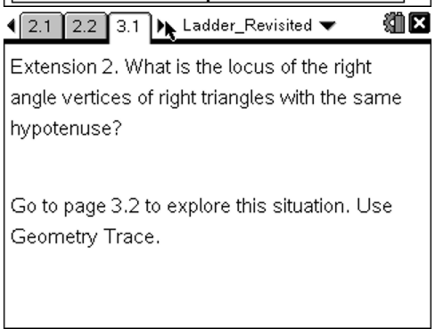
The Ladder Revisited
Student TI-Nspire Document: *Ladder_Revisited.tns*




1.8 1.9 1.10 Ladder_Revisited
Statements
Reasons


1.9 1.10 2.1 Ladder_Revisited
Extension 1. What if the painter is standing at any other point on the ladder? What is the trajectory (locus) of this new point?
Go to page 2.2 to explore this situation.


1.10 2.1 2.2 Ladder_Revisited
P
drag me
Instructions: Click Menu → 5: Trace. Choose 3: Geometry Trace. Click on point P. Then drag the point along the line and observe the trace.


2.1 2.2 3.1 Ladder_Revisited
Extension 2. What is the locus of the right angle vertices of right triangles with the same hypotenuse?
Go to page 3.2 to explore this situation. Use Geometry Trace.