Continuous and Differentiable Functions Exploration

Objective: Given a hybrid function, make the function continuous at the boundary between the two branches. Then make the function differentiable at this point.

Let *f* be the function defined by

$$f(x)=\begin{cases} x+1, & x<2 \\ k(x-5)^2, & x\geq 2 \end{cases}$$

where *k* is a constant.

1. Sketch the graph of *f* for $k=1$.

2. Function is discontinuous at $x=2$. Explain what it means for the function to be discontinuous.

3. Find $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$. The second limit will be in terms of *k*. What must be true of these two limits for *f* to be continuous at $x=2$?

4. Find the value of *k* that makes *f* continuous at $x=2$. Sketch the graph of *f* for this value of *k*.

5. The graph in part 4 has a cusp at $x=2$. Cusp comes from the Latin *cuspis*, meaning a point or a pointed end. Why is it appropriate to use the word cusp in this context?

6. Suppose someone asks, 'Is *f(x)* increasing or decreasing at $x=2$ with *k* as in part 4?" How would you have to answer that question? What, then, can you conclude about the derivative of a function at a point where the graph has a cusp?

7.Sketch the graph of $f'(x)$ for the value of *k* you found in part 4.

8. For two graphs to join smoothly, the gradients on both sides have to be equal. Let's define a new function g:

$$g(x) = \begin{cases} x+1, & x<2 \\ ax^2+bx, & x \geq 2 \end{cases}$$

Find the values of *a* and *b* so that both graphs join smoothly.

9. Sketch the graph of $g(x)$ and $g'(x)$.