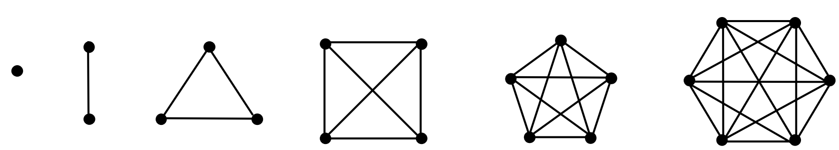
NUMB3RS Activity: A Party of Six
Episode: “Protest”

Topic: Graph Theory and Ramsey Numbers **Grade Level:** 8 - 12
Objective: To see how a complete graph with edges of two colors can be used to model acquaintances and non-acquaintances at a party.
Time: About 30 minutes
Materials: Red and blue pencils or markers, paper

Introduction
In “Protest,” Charlie uses social network analysis to try to identify modern day criminals by tracing them through past social relationships. Students might wonder how mathematics could possibly be useful in the context of social relationships. This activity exposes students to a simple social problem, and illustrates how the branch of combinatorics known as *graph theory* can be used to solve it. Interestingly, the generalizations of this “simple” problem can quickly become very difficult (see the *Extensions* page of this activity).

Discuss with Students
Algebra students are probably familiar with graphs of functions, but there are other kinds of graphs that are useful in mathematics. In general, a graph is a picture that extends our understanding of a problem by modeling it visually. Previous *NUMB3RS* activities have exposed students to histograms, scatterplots, time graphs, probability trees, and Voronoi diagrams – all of which are useful visualizations of mathematics. Previous activities have also used the graphs that we will consider here. Their properties are the subject of graph theory.

To a graph theorist, a graph is a collection of points (called *vertices*) and lines (called *edges*) connecting some, but not necessarily all, of the points. The *complete* graph on n vertices, denoted by K_n, consists of n vertices connected by all possible edges. The first six complete graphs are shown below:



In this activity we use complete graphs to model relationships (acquaintances or non-acquaintances) among people at a party.

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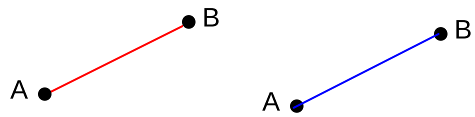
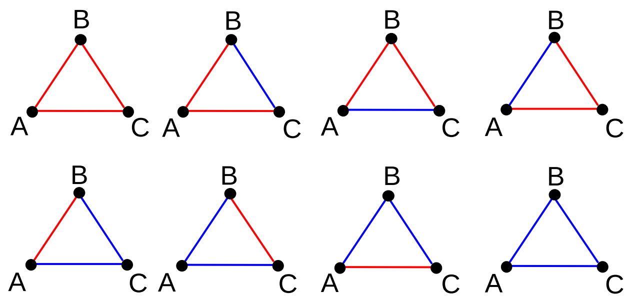
Student Page Answers:

1.



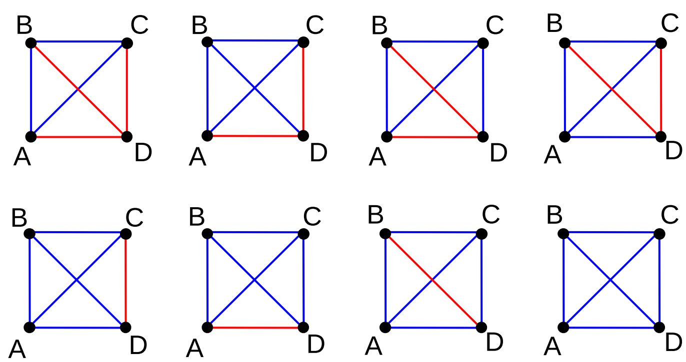
2. *$2^3 = 8$*

3.



4. *All blue: All three people know each other. All red: No two know each other.*

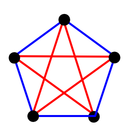
5. *A complete 4-graph has 6 edges, so $2^6 = 64$ different colorings are possible.*



6. *Eight. In fact, just switch colors in the eight graphs in answer 5*.

7. *Look for a triangle with edges of the same color.*

8. *One possible solution:*



9. *A complete 6-graph has 15 edges, so there are 2^{15} = 32,768 colorings!*

10. *HEXI can never end in a draw; either blue or red will win*.

11. *<u>Case two</u>: There are at least three other people who are not acquainted with A. If any two of those three are not acquainted with each other, these two and A form three mutual non-acquaintances. If all of those three are acquainted with each other, they form three mutual acquaintances.*

Name: ________________________________ Date: _______________

NUMB3RS Activity: A Party of Six

In “Protest,” Charlie uses a relatively new application of mathematics called social network analysis (SNA) to identify modern criminals by tracing their past associations with like-minded friends and co-workers. In this activity, you will use the branch of combinatorial mathematics called *graph theory* to represent acquaintances and non-acquaintances at a party, a fairly simple problem that leads to questions in *Ramsey Theory* that continue to challenge researchers today.

We will use dots (called *vertices*) to represent people at a party. We will connect two vertices with a blue line segment (or *edge*) if the two people know each other. If they do not know each other, we will connect them with a red edge. Each such diagram of vertices and edges is called a *graph*. A graph is called *complete* if every pair of vertices is connected by an edge. Because every pair of guests will be connected by either a red edge or a blue edge, all of our "party" graphs will be complete.

Parties of Two and Three

1. If two people (A and B) are at a party, there are only two possibilities: either A and B know each other, or A and B do not know each other. Draw the two possible graphs below.

● B ● B

A ● A ●

2. The complete graph representing a party of three (A, B, and C) is a triangle. If each edge can be either of two colors, how many possible 3-person party graphs are there? _________

3. Draw all of the possible 3-person party graphs for A, B, and C below.

4. What does a triangle graph with all red edges represent? _____________________________
What does a triangle graph with all blue edges represent?_____________________________

A Party of Four

5. There are 64 possible 4-person party graphs for guests A, B, C, and D (Why?), but you will not be asked to draw them all. Instead, draw the 8 possible 4-person party graphs in which A, B, and C all know each other. We say A, B, and C are *mutual acquaintances*.

6. How many 4-person graphs can be drawn in which A, B, and C are *mutual non-acquaintances*?______________________________

7. There is a simple way to spot three mutual acquaintances or three mutual non-acquaintances in any two-colored complete graph. What do you look for?

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A Party of Five

8. It is actually possible to color the edges of a 5-person party graph in such a way that there are neither three people that are mutual acquaintances nor three people that are mutual non-acquaintances. Can you do it? Remember, every pair of vertices must be connected with an edge of one color or the other.

A Party of Six

9. It is an interesting fact that every party of 6 people *must* contain either three mutual acquaintances or three mutual non-acquaintances. You will not be asked to draw all the possibilities, but you should be able to *count* them.

How many edges does a complete 6-graph have?________________________

If each edge can be either of two colors, how many 6-person party graphs are possible for guests A, B, C, D, E, and F?______________________________

10. The game of HEXI is played by drawing six vertices in a hexagonal array. Player A connects two vertices with a blue edge, player B connects two vertices with a red edge, and so on. The first player to complete a triangle of his color *loses* the game. What is the implication of the “interesting fact” mentioned in Question #9 for the game of HEXI?

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11. Challenge The proof that every party of 6 people must contain either three mutual acquaintances or three mutual non-acquaintances is an application of simple logic, but two separate cases must be considered.

Begin with person A. The other five people must include at least three who are acquainted with A or at least three who are not acquainted with A.

<u>Case One</u>: There are at least three other people who are acquainted with A.

- If any two of those three are acquainted with each other, these two and A form three mutual acquaintances.
- If none of those three are acquainted with each other, they form three mutual non-acquaintances.

Complete the proof by defining and analyzing the other case.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Social Networks and Visualization

- A social network visualization package called Vizster was used to demonstrate the social network graphics for the "Protest" episode. The computer science post-doctoral students who developed Vizster at the University of California at Berkeley have created a video demonstration of the software. This video can be downloaded from the Web site below.
  http://jheer.org/vizster/
- You can find out more about SNA in the activity **It's All Connected**, which accompanied the *NUMB3RS* episode "The O.G." This activity can be downloaded for free from the Web site below.

  http://education.ti.com/educationportal/activityexchange/activity_detail.do?cid=us&activityid=6447

  Act2_ItsAllConnected_TheOG_final.pdf

Ramsey Numbers

The *Ramsey Number R*(m, n) gives the minimum number of people at a party that will guarantee the existence of either m mutual acquaintances or n mutual non-acquaintances.
If you completed Questions 8 and 11 in the activity above, you constructed a proof that $R(3, 3) = 6$. Ramsey numbers are named after Frank Plumpton Ramsey (1903-1930), whose work attracted the interest of many mathematicians, notably the renowned Paul Erdös. *Ramsey's Theorem* shows that there is a Ramsey number $R(m, n)$ for all pairs m and n, but it does not give a formula for finding them. Thus, the search for Ramsey numbers (like the search for large primes) is an interesting and ongoing mathematical quest.

- You can read about Ramsey numbers and see a list of those that are known at **http://mathworld.wolfram.com/RamseyNumber.html**.
- Ivars Peterson writes an online column called *Math Trek* for the Mathematical Association of America. Check out his article "Party Games" at the Web site below.
  http://www.maa.org/mathland/mathtrek_12_6_99.html.
- Ivars Peterson credits Wolfgang Slany at the Technical University of Vienna, Austria with the invention of the game HEXI. You can play a Java version of the game against the computer at **http://www.dbai.tuwien.ac.at/proj/ramsey/**.
- Even with supercomputers, finding Ramsey numbers is not an easy task. It has been known since 1955 that $R(4, 4) = 18$, so the smallest number of people at a party that will guarantee either four mutual acquaintances or four mutual non-acquaintances is 18. The exact value of $R(5, 5)$ is as yet unknown, although it is known to lie between 43 and 49. We also know that $R(6, 6)$ lies between 102 and 165. The difficulty of finding the exact value of either one is suggested by the following whimsical quote by Paul Erdös:

  "Imagine an alien force vastly more powerful than us landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. Then we should attempt to destroy the aliens."

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