Welcome to the MATRIX

Transforming 2D Space

Teacher Notes & Answers

7 8 9 10 **11** **12**

TI-Nspire™ Investigation Student 45 min

Teacher Notes:

The purpose of this activity is to introduce students to matrix transformations visually and intuitively. It is HIGHLY recommended students watch the following videos created by Grant Sanderson (Three Blue One Brown). These videos are a part of a full series on Linear Algebra that provide a wonderful look at transformations, linear dependence and independence, determinant of a matrix, dot and cross products of vectors and much more. The TI-Nspire file and activities are designed to help connect the conceptual understanding with relatively simple practice and engagement.

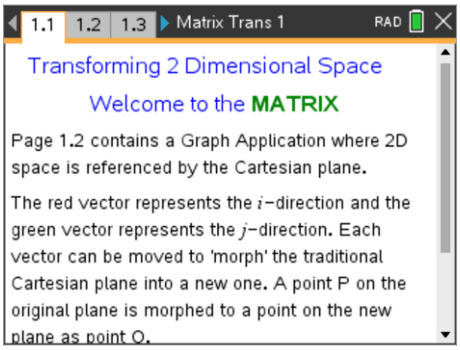
QR codes for these videos are included on the student handout.

- Vectors Chapter 1, Essence of Linear Algebra: https://youtu.be/fNk_zzaMoSs
- Linear Combinations, span and basis vectors: https://youtu.be/k7RM-ot2NWY
- Linear Transformations and Matrices: https://youtu.be/kYB8IZa5AuE

Transforming 2-Dimensional Space

Open the TI-Nspire file: Matrix Trans 1

Page 1.1 contains a brief introduction. Read the instructions then navigate to page 1.2.



The red vector is a transformation of the unit vector: $\underset{\sim}{\hat{i}}$.

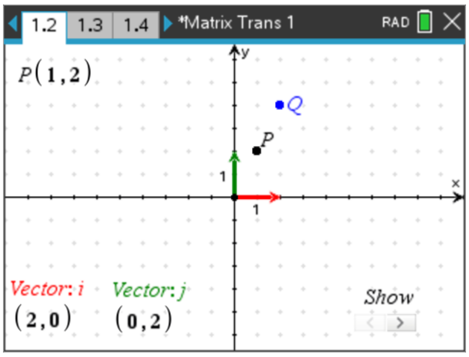
The green vector is a transformation of the unit vector: $\underset{\sim}{\hat{j}}$

Point P exists on the Cartesian plane. The coordinates of point P are displayed in the top left corner of the screen.

Point Q is the image of P using the transformation created by the combination of the red and green vectors.

The coordinates for the red and green vectors are displayed in the bottom left corner of the screen. These two vectors can be used to transform the plane.

In the screen shown opposite, the plane has been dilated by a factor of 2 in both the x and y direction.



Question: 1.

Suppose point P is moved to (2, 3). Using the same transformation as above, what would be the location of Q?

Answer: Point Q: (4, 6) **Note**: The intention is for students to determine this *intuitively*.

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Question: 2.

Suppose point P is at (x, y) and undergoes the same transformation as above, such that point Q is (-6, 8). What would be the coordinates of point P?

Answer: Point P: (-3, 4) **Note**: The intention is for students to determine this *intuitively*.

Question: 3.

Change vector $\underset{\sim}{i}$ to a length of 4 units, in the positive x direction (only) and vector $\underset{\sim}{j}$ to a length of 3 units, in the positive y direction (only). With point P at (1, 2), what are the coordinates of point Q?

Answer: Point Q: (4, 6) **Note**: The intention is for students to determine this *intuitively* or using the TNS file.

Question: 4.

Change vector $\underset{\sim}{i}$ to a length of 2 units, in the positive x direction (only) and return vector $\underset{\sim}{j}$ to a unit length (1), in the positive y direction (only). Point P can be placed in each of the locations below.

a) For each location, determine the corresponding coordinates of the transformed point: Q.

P: (x, y)	(-2, 4)	(-1, 1)	(0, 0)	(1, 1)	(2, 4)
Q: (x, y)	(-4, 4)	(-2, 1)	(0, 0)	(2, 1)	(4, 4)

b) The location of each point P (above) could be described by a function: $y = x^2$.

Study the points generated for Q and determine an appropriate function to describe the location of these transformed points.

Answer: Equation: $y = \dfrac{x^2}{4}$

Teacher Note: The intention is for students to determine this *intuitively*, by studying the points. Students should be encouraged to graph their answer. Students may graph: $y = (½)x^2$, and realise that this equation is not correct, that too is perfectly okay.
At this stage, the idea is for students to explore numerically, not algebraically.

At the conclusion of this question, return vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ to 2 units each in the x and y directions respectively.

The world according to Q.

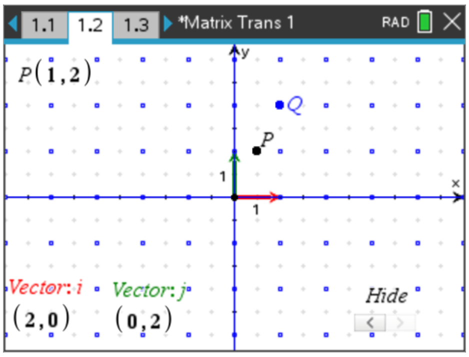
The show/hide toggle in the bottom right section of the screen allows you to view the transformed plane, "Q's World".

The blue points show how the original 1 x 1 grid has been stretched in both the x and y direction, each by a factor of 2, just like a scaled drawing.

The coordinates of point Q on the *transformed plane* are: (1, 2), it's position relative to the transformed plane is the same as P on the original plane. Point Q is the image of point P under the transformation of the plane.

In mathematics we are generally interested in the coordinates of Q, relative to the original plane. In reference to the original plane, Q is located at the point: (2, 4).

It is useful to remember that in Q's world, (the transformed plane), it's relative position is unchanged.



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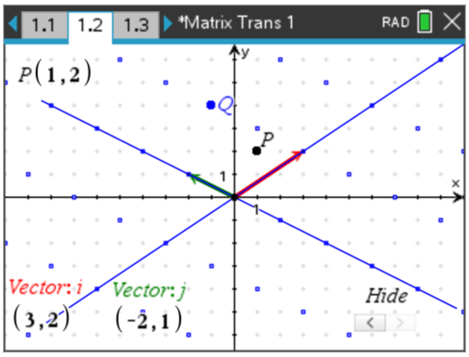
Vectors *i* and *j* can *skew* the plane. In the diagram shown here, vector *i* has been moved to (3, 2) and vector *j* to (-2, 1). Relative to the skewed plane, Q is still located at: (1, 2).

Relative to the original cartesian plane, Q is located at: (-1, 4)

The transformation can be summarised by a matrix that contains the two vectors:



vector $\underset{\sim}{i}$	vector $\underset{\sim}{j}$	Transformation Vector
$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$

When the transformed plane is displayed (toggle on), content connected to the *i* and *j* vectors incorporates many calculations slowing the calculator down … .

The calculator will respond quicker if the transformed plane is hidden while vectors are being moved.

Question: 5.

The point P can be expressed as a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

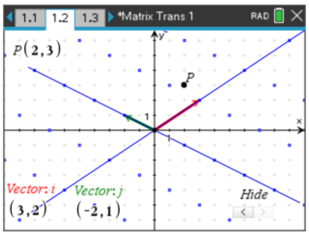
a) Determine the product of the transformation matrix: $\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$ and point P $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$; comment on the result.

Answer: $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ Comment: This represents the coordinates of point Q.

b) Supposed point P is moved to (2, 3) and undergoes the same transformation, determine the new location of point Q (the image of P) under this transformation using matrices and the interactive TI-Nspire diagram.

Answer: $\begin{bmatrix} 0 \\ 8 \end{bmatrix}$

Comment: Students manipulate P and the two vectors, Q is just off screen.



c) Point P is located at $\begin{bmatrix} x \\ y \end{bmatrix}$ and undergoes the same transformation as above such that Q is located at (9, -1). Determine the location of point P.

Answer: Students may obtain the answer informally working backwards, or experimentally moving point P (students cannot move point Q), however by this stage students should be using matrix multiplication and 'inverse' matrices, a more expedient calculation:

$$\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

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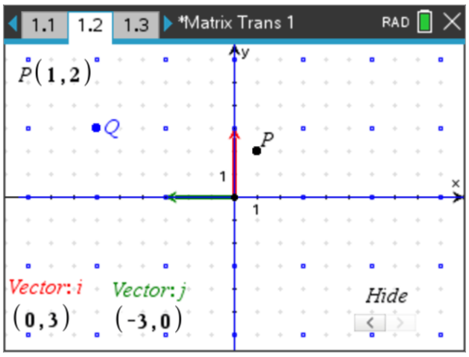
In the image shown here, the $\underset{\sim}{\hat{i}}$ vector has been moved onto the y axis and dilated by a factor of 3. At the same time the $\underset{\sim}{\hat{j}}$ vector has been moved to the negative x axis and also dilated by a factor of 3.

This transformation can be described numerically using the matrix:

$$\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$$



Question: 6.

a) Determine the location of point Q using the transformation matrix (given above).

Answer: $\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

b) Compare the distances: $|OP|$ and $|OQ|$.

Answer: $|OP| = \sqrt{5}$ and $|OQ| = \sqrt{45} = 3\sqrt{5}$, therefore: $|OQ| = 3|OP|$

c) Describe the transformation geometrically.

Answer: Point P is rotated counter-clockwise 90° and then dilated away from the origin by a factor of 3.

Question: 7.

P(x, y) is transformed such that $Q\left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ and Q is located at: (-7, -1):

a) Determine the location of P.

Answer: $\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} -7 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ Encourage students to check their answer using the TNS file.

b) Suppose point P moves along the line: $y = x + 1$, describe the corresponding location of Q.

Answer: By experimentation Q moves along the line: y = -1

Teacher Note: If students are going to complete the other activities in this series, don't be tempted to show students the algebraic approach for this question. The path for P was chosen to create a very simple relationship for Q, provided students move P accordingly. This is however an opportune time to reference "Linear algebra" and "Linear transformations" referring to the fact that under transformations, linear functions will remain as linear functions.

c) Suppose point P moves along the x axis, describe the corresponding location of Q.

Answer: $y = -x$

Teacher Note: If students have the transformed plane active, the answer should become remarkably obvious. This task specifically refers to "Q's World", so if P is moving along the x axis, then in Q's World, it will move along the transformed x axis. Students can be reminded that the i-direction has been changed to (-1, 1), this is the new 'x direction' and therefore the new 'x-axis'.

Question: 8.

Suppose P(x, y) is transformed such that $Q\left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

a) Move point P and describe the corresponding locations for Q.
Answer: Point Q moves along the line $y = 2x$.

b) Explain what has happened geometrically.
Hint: Switch on the transformed plane view.
Answer: The cartesian plane (2D) has been transformed into a single dimension (line).

c) Point Q is located at (2, 4). Determine the location(s) for point P.
Answer: If students attempt to solve this problem using matrices, they will get an error: "singular matrix" as the determinant is equal to zero. Students can solve the problem using the TI-Nspire file.
P can be anywhere on the line: $y = x - 2$.

Teacher Notes: Encourage students to think about the concept that all points on the line $y = x - 2$ have been collapsed into a single point. This is also a great opportunity to discuss the fact that the determinant is zero. Later in this series of activities, students need to re-visit the determinant to see how it impacts properties like area.

d) Identify another transformation that would have a similar impact.
Answer: Answers will vary. Students may establish the result experimentally or through realisation that any matrix with a determinant = 0 will cause the same collapsing of the 2D plane into one dimension. (1D)

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