ANALYZING RELATED RATES

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Key Topic: Applications of Derivatives

Abstract:

This activity is designed for students who have had a brief introduction to related rates and are now ready to attempt to solve some real world applications. The activity contains a review of the basic method of solving related rate problems, and then it concentrates on methods needed to set up and analyze more complicated problems.

The main emphasis of this activity is on how to set up the problem. This is usually the hardest thing for students. Although standard textbook problems will be used in this activity, students will not simply be asked to solve the problem. They will be guided by a series of questions which requires them to first analyze the problem before solving the problem.

Prerequisite Skills:

- A brief introduction to related rates.
- Knowledge that the derivative can be used to find rate of change.
- Ability to find derivatives using implicit differentiation.
- Basic understanding of geometry. In particular:
    - The Pythagorean theorem
    - Similar triangles
    - Basic formulas for area and volume
- Basic understanding of right triangle trigonometry.

Degree of Difficulty: Moderate to hard

Needed Materials: TI-89

NCTM Principles and Standards:

- Content Standards – Algebra
    - Represent and analyze mathematical situations and structures using algebraic symbols
    - Use mathematical models to represent and understand quantitative relationships
    - Draw a reasonable conclusion about situation being modeled

- Content Standards - Geometry
    - Use geometric modeling to solve problems

- Process Standards
    - Representation
    - Connections
    - Problem Solving

ANALYZING RELATED RATES

Many years ago I went to a talk on memory which was given by Linda Palm, a psychologist at Coastal Carolina University. She said something which radically changed the way in which I have approach teaching word problems to my students. What she said was:

> **"Students can't do word problems because by the time they have finished reading the problem, they have forgotten what they have read."**

So to combat this problem of forgetting what you have read, I suggest that when solving *any* mathematical problem it is best to first do the following:

1. Define your variables, if they aren't defined for you.
2. Draw a picture, if appropriate.
3. Identify what is given.
4. Identify what you need to find.

Sometimes, step 1 and/or 2 may not be needed. But steps 3 and 4 are *always* needed. If you are good about remembering things, you may not need to actually put these steps in writing. But if you are leery of word problems, I suggest that you do put them in writing.

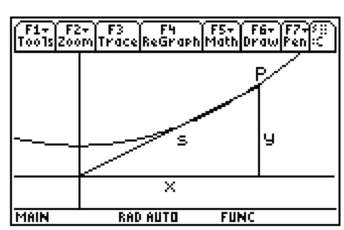
In solving related rate problems, the next steps are usually to:

5. Find an equation which relates *only* the variables for which you are given information about their rate of change or for which you must find their rate of change. This step may require the use of geometry and/or trigonometry.
6. Use implicit differentiation to take the derivative of both sides of this equation with respect to the required variable (which is usually the "time" variable). If the values of any rates were given in the problem, they should be substituted into this equation at this time.
7. If requested, evaluate the rate you where asked to find using any values which the problem specifies for the variables.

EXAMPLE: A point is moving on the graph of $y = x^2 + 1$ in such a fashion that $\frac{dx}{dt} = 2$ centimeters per second. Find the rate of change of the distance between the origin and the moving point when $x = 1$.

Step 1. $s =$ the distance between the origin and the moving point *P*.
$x =$ the *x*-coordinate of the moving point *P*.
$y =$ the *y*-coordinate of the moving point *P*.

Step 2. See figure at the right.



Step 3. ***Given:*** $y = x^2 + 1$

$$\frac{dx}{dt} = 2 \text{ centimeters per second}$$

Step 4. ***Find:*** $\frac{ds}{dt}$ when $x = 1$

Step 5. Since we are given information about $\frac{dx}{dt}$ and are asked to find $\frac{ds}{dt}$, we need to find an equation which relates ***only*** the variables *x* and *s*.

By the Pythagorean theorem, $s^2 = x^2 + y^2$. But this equation relates more than just the 2 variables *x* and *s*! We have to get rid of the *y*! How do we do this?

If you don't immediately see how to do this, then you have forgotten what you read in the problem. But this isn't a problem because steps 1 through 4 forced you to take good noted. In step 3 you are given that $y = x^2 + 1$.

So $s^2 = x^2 + y^2 = x^2 + (x^2 + 1)^2$. This simplifies to $s^2 = x^4 + 3x^2 + 1$.

Step 6. Since we are asked to find rates with respect to time *t*, we take the derivative of both sides of this equation with respect to *t*. This yields:

$$2s\frac{ds}{st} = 4x^3\frac{dx}{dt} + 6x\frac{dx}{dt} + 0$$

$$\frac{ds}{dt} = \frac{(2x^3 + 3x)\frac{dx}{dt}}{s}$$

Since we were given that $\frac{dx}{dt} = 2$ centimeters per second

$$\frac{ds}{dt} = \frac{2 \cdot (2x^3 + 3x)}{s} \text{ centimeters per second}$$

Step 7. We are asked to evaluate this when $x = 1$. But this will give us only the numerator. That is: $\frac{ds}{dt} = \frac{10}{s}$. How do we find *s*, the denominator? Well back there at the end of step 5 we found that $s^2 = x^4 + 3x^2 + 1$. So when $x = 1$, $s = \pm\sqrt{5}$. Which sign do we use? Since $s =$ distance, and distance is always positive, $s = +\sqrt{5}$. Hence $\frac{ds}{dt} = \frac{10}{\sqrt{5}}$ centimeters per second.

HELPFUL HINTS:

1. Units of measurement are not a hindrance, they're a help. When you are given information about something, the units attached to that "something" help you figure out what that "something" is. To put it bluntly, distance and length are measured using feet, centimeters, etc. Area uses square feet, sq. cm., etc. And volume is measured in cubic feet, cu. cm., etc.

   When dealing with rates, you know that $\frac{dw}{dt}$ denotes the rate of change in *w* over time *t*. So it's units should be $\frac{\text{units used to measure } w}{\text{units used to measure } t}$.

   For example, if you are given that water is being pumped into a pool at a rate of 0.25 cubic meters per minute, then the units "cubic meters" tell you that you are looking at the rate of change in the volume *V* of water in the pool. So this piece of information tells you that $\frac{dV}{dt} = 0.25$ where $V =$ the volume of the water in the pool.

2. Not all rates are positive! If w is increasing over time, then $\frac{dw}{dt}$ is positive. But if w is decreasing, then $\frac{dw}{dt}$ is negative. For example, if you are given that water is being drained *out* of a pool at a rate of 0.25 cubic meters per minute, then the volume *V* of water in the pool is decreasing. So $\frac{dV}{dt} = -0.25$.

Now that you know the basics, try the following problem by answering the accompanying questions.

PROBLEM: A ladder 25 feet long is leaning against the side of a house. The ladder begins to slip down the side of the house at a rate of 2 feet per second. How fast is the base of the ladder moving away from the house when the base of the ladder is 24 feet from the house.

QUESTION 1. Draw a picture of the scenario described in the problem and identify your variables.

QUESTION 2. Identify what you are given and what you must find.

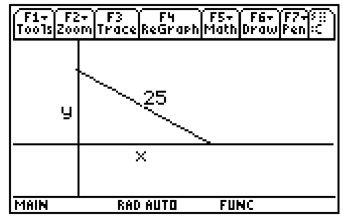
QUESTION 3. Find an equation which relates *only* the variables for which you are given information about their rate of change or for which you must find their rate of change.

QUESTION 4. Use implicit differentiation to take the derivative of both sides of this equation with respect to the required variable. If appropriate, simplify this equation.

QUESTION 5. If the values of any rates were given in the problem, substitute them into this equation at this time.

QUESTION 6. Find what you were asked to find.

SOLUTIONS:



1. $x =$ the distance of the base of the ladder from the house.

   $y =$ the distance of the ladder from the ground.

2. **Given:** $\dfrac{dy}{dt} = -2$ cm. per sec. (Note: since y is decreasing, $\dfrac{dy}{dt}$ is negative.)

   Find: $\dfrac{dx}{dt}$ when $x = 24$.

3. $x^2 + y^2 = 25^2$

4. $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0 \quad \Rightarrow \quad x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = 0$

5. Since $\dfrac{dy}{dt} = -2$, the equation above becomes $x\dfrac{dx}{dt} - 2y = 0$.

6. To find $\dfrac{dx}{dt}$ when $x = 24$, we first need to know what y is when $x = 24$.
   Using the answer to #3, we see that $24^2 + y^2 = 25^2$. This gives us $y = 7$.
   Substituting this into the answer to # 5 gives $24\dfrac{dx}{dt} - 2 \cdot 7 = 0$. From this we find that
   $\dfrac{dx}{dt} = \dfrac{14}{24} = \dfrac{7}{12}$ feet per second.