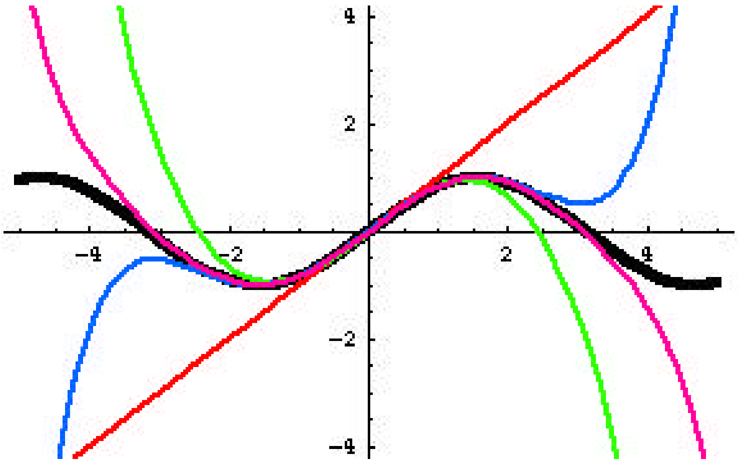
Exploring Taylor Series

Teachers Teaching with Technology
16-18 March 01



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Sequences and Series
on a calculator

Example: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$

To show the sequence $a_k = \frac{(-1)^{k+1}}{k^2}$ and the sequence of partial sums $S_n = \sum_{k=1}^{n} a_k$:

Sequence Mode

TI-83	TI-89

`nMin = 1`
`u(n) = (-1)^(n+1)/n^2`
`u(nMin)=`
`v(n) = v(n-1)+ (-1)^(n+1)/n^2`
`v(nMin) = 1`
	`u1 = (-1)^(n+1)/n^2`
	`ui1=`
	`u2 = u2(n-1)+(-1)^(n+1)/n^2`
	`ui2 = 1`

Notes:

(1) Since u is given explicitly, u(*n*min) = ui1 should be left blank.

(2) v(*n*min) = ui2 should be equal to the value of $a_1 = S_1$ for your sequence.

(3) The x-window should agree with the n's.

(4) The y-window should appropriate for the values of terms of your sequence and series.

Parametric Mode

TI-83	TI-89

X_{1T} = T
Y_{1T} = (-1)^(T+1)/T^2
X_{2T} = T
Y_{2T} = sum(seq(Y_{1T},T,1,T))
	`xt1 = t`
	`yt1 = (-1)^(t + 1)/t^2`
	`xt2 = t`
	`yt2 = sum(seq(yt1(t),t,1,t))`

Notes for skeptics:

(1) Here, the t and x windows should be the same, or at least mildly close.

(2) t-step should equal 1.

(3) The sum/sequence can take a while to calculate, depending upon the sequence you choose and how many terms are to be considered.

(4) In this mode, it is not necessary to type the sequence twice. And twice can be annoying. Also, the necessary variables are easier to hit – no alpha or 2nd key required.

TI – 89

F3 → 9: taylor(

taylor(expr, var, order [, point]) where "point" is optional, assumed to be 0 unless stated otherwise

Ex 1: Taylor series for sinx, degree 7, centered at a = 0

taylor(sin(x), x, 7)

Ex 2: Taylor series for lnx, degree 5, centered at a = 1

taylor(ln(x), x, 5, 1)

Wonderful for animations:

Graphing Calculator (software) www.pacifict.com

For example, enter the functions y = sinx and $y=\sum_{k=0}^{n}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$ and set the slider to n: 0 → 10 with 10 steps.

Tails with *Mathematica* (or something close)

One interesting way of sensing how fast an infinite series converges is to sum out a ways, say to some finite value k, and compare this to the infinite sum.

Consider the three convergent series $a_n=\frac{1}{e^n+1}$, $b_n=\frac{1}{n^3+1}$, and $c_n=\frac{1}{n^2+1}$. Find the infinite sum. Then find the sum from 1 to k, where k is some constant. How large does k have to be for each series in order to get accuracy to 4 decimal places? You may be surprised.

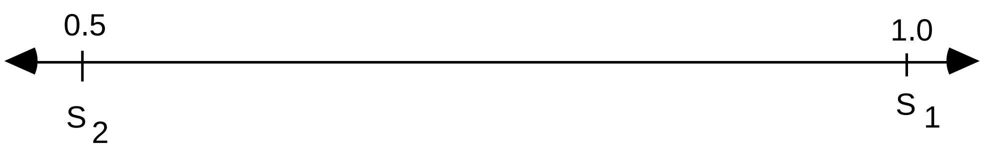
Alternating Series

Defn: If $a_n > 0$, an <u>alternating series</u> is a series of the form

$$a_1 - a_2 + a_3 - a_4 + \ldots \quad \text{OR} \quad -a_1 + a_2 - a_3 + a_4 - \ldots$$

Ex: Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots,$ the alternating harmonic series.

Find $S_1, S_2, S_3, \ldots, S_{10}$ to two decimal places, plotting each on the line below. (This has been started for you.)



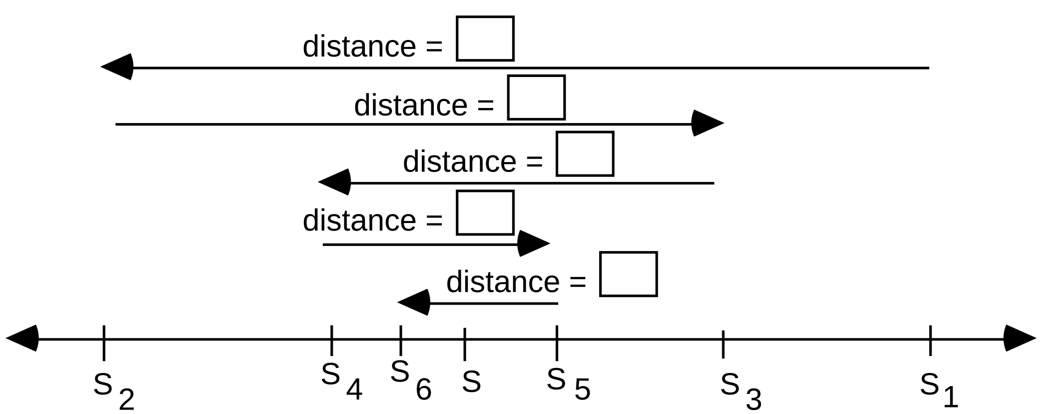
Describe the pattern of the S_n.

Do you think the series converges?

Where is S? (Write an inequality involving S.)

Approximating Alternating Series

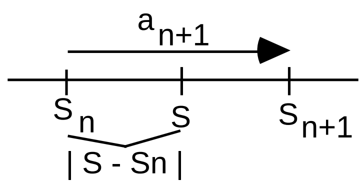
Consider the alternating series $a_1 - a_2 + a_3 - a_4 + \ldots$ with $a_n \to 0$. Remember that all $a_n > 0$. Fill in the distances (positive) given by each arrow.



Usually, it is impossible for us to calculate S directly. Therefore, we wish to approximate S by using one of the S_n. For example, let's use S_5. The next real question or problem is to find out just how good S_5 is as an approximation. In other words, we need to know how close S_5 is to S.

We can express this distance as $| S - S_5 |$. More generally, we need to look at the size of $| S - S_n |$. Since the S_n oscillate in smaller and smaller steps around S, then S is between S_n and S_{n+1} for all n.

Consider this arbitrary stage of the diagram above.



Here, we can see that $| S - S_n | <$ ________. More specifically, we have

$| S - S_3 | <$ ________, $| S - S_5 | <$ _________, and $| S - S_{20} | <$ _________.

Ex: Given the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$,

(a) if we approximate S by using 10 terms, what will be the magnitude of the error? In other words, how large is $| S - S_{10} |$?

Find S_{10} and write an inequality about S.

(b) How many terms are necessary to be sure that the error is less than .01 ?

Conditional Convergence

Here's a neat little "proof" of 1 = 2.   (This is at a slightly higher level than the one often seen at the Algebra I level.) Discuss the concept of conditional convergence before doing this "proof." This should convince students that these series must not be approached too casually!

Let S $= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \frac{1}{16} + \ldots$

$$= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \frac{1}{9} - \ldots$$

$$= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} + \left(\frac{1}{9} - \frac{1}{18}\right) - \frac{1}{20} + \ldots$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \frac{1}{18} - \frac{1}{20} + \ldots$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \ldots\right)$$

$$= \frac{1}{2} S$$

$$S = \frac{1}{2}S \qquad 1 = \frac{1}{2} \qquad 2 = 1$$

Theorem: Any conditionally convergent series can be rearranged to add up to any given number.

Graphing Pairs

(1) Use a window with x [-3, 3] and y [-3, 3] to plot the graphs of the following functions:

$$f(x) = \sin x \quad \text{and} \quad g(x) = x\,(\cos x)^{1/3}$$

Make a careful sketch.

Describe what happens. (Where are they close? How close?)

(2) Use a window with x [-2, 5] and y [-1.5, 1.5] to plot the graphs of the following functions:

$$f(x) = \sin x \quad \text{and} \quad g(x) = -0.4177x^2 + 1.3122x - 0.0505$$

Make a careful sketch.

Describe what happens.

(3) Use a window with x [-4, 4] and y [-2, 2] to plot the graphs of the following functions:

$$f(x) = \sin x \quad \text{and} \quad g(x) = \frac{60x - 7x^3}{60 + 3x^2}$$

Make a careful sketch.

Describe what happens.

Series 1

(1) Let $f(x) = \ln(1 + x)$ and let $g(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Set $f(0) = g(0)$, and set the first three derivatives evaluated at x = 0 equal to each other in order to find the values of the a_i and the polynomial g(x). (In other words, set $f^{(k)}(0) = g^{(k)}(0)$ for k = 0, 1, 2, 3.)

(2) Find both $f(a)$ and g(a) for each of the following values of a. Are they very close to each other? When are the values closer?

	$f(a)$	g(a)
a = 0.5		
a = 0.1		

(3) Use your calculator to graph both functions f and g. Try different windows to get a good view and copy the graph below. On what interval does g seem to be useful as an approximation of f?

(4) Now sketch a graph of $R(x) = | f(x) - g(x) |$. This represents the remainder, or the distance between the function f and its approximation g. Describe the graph and what this says about g as an approximation.

(5) Extend g by continuing the pattern of the coefficients of g to form an infinite series which will approximate the function f. Use an appropriate test to determine the open interval for which this series converges.

Check the endpoints of your interval to determine whether the series will also converge at those values.

(6) If we use only the three terms of our polynomial g — the first three terms of our infinite series — and we use x = 0.75, find an upper bound for the error. Find S_3 and use this to write an inequality for S, the infinite sum.

(7) Again using these three terms and assuming $x > 0$, what are the possibilities for x if the error is to be less than 0.005?

(8) If $0 < x < 0.5$, how many terms of the series do you have to use in order to get an error which is less than 0.001 ?

Series 2

Assuming we know something about $f(x) = \cos x$ ...

We found a polynomial which can be extended to give

$$P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Let $P_n(x)$ refer to the polynomial approximating the function f which has "order of contact n" at a point x = a. That is to say, $P_n^{(k)}(a) = f^{(k)}(a)$ for k = 0, 1, 2, 3, ... , n, and here we have a = 0.

For example, with $f(x) = \cos x$, we have

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad \text{and} \quad P_7(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

Note that “n” can cause problems. Does it mean the nth term? the order of contact? the value of n in the summation? Just be careful!

(1) If we choose $P_4(x)$, giving us 3 (non-zero) terms, evaluate $f(x)$ and $P_4(x)$ for the following values of x.

	$f(x)$	$g(x) = P_4(x)$
x = 1		
x = 0.5		

For what values of x does P_4 seem to be a good approximation of f?

(2) Again using 3 terms, let x = 1. How big will the error be? This time, do this two ways and note the difference.

(a) Find $|\cos 1 - P_4(1)|$. (Simply evaluate cos1 on your calculator.)

(b) Use the alternating series error approximation.

(3) Still using 3 terms of P, what are the possibilities for x if the error is to be less than 0.00005?

Set the window on the calculator so that $-7 \le x \le 7$ and $-2 \le y \le 2$.

(4) Plot $P_4(x)$ and $f(x)$. Sketch the graphs below.

For what values of x does P_4 seem to be a good approximation of f ?

(5) Plot $P_6(x)$ and $f(x)$. Sketch the graphs below.

For what values of x does P_6 seem to be a good approximation of f ?

(6) Plot $P_{10}(x)$ and $f(x)$. Sketch the graphs below.

For what values of x does P_{10} seem to be a good approximation of f ?

(7) Plot $P_{14}(x)$ and $f(x)$. Sketch the graphs below.

For what values of x does P_{14} seem to be a good approximation of f ?

(8) Based on these graphs, can you make any guesses about the values of x for which the infinite series P converges?

Do you think that there is a value of n such that $P_n(x)$ will have an error less than 0.0001 for all x in the interval $-1000 \le x \le 1000$? Why or why not?

(9) Determine the values of x for which the infinite series converges by using an appropriate test.

(10) Compare this result to the intervals found for $y = \ln(1 + x)$ and $y = \tan^{-1}x$.

Can you think of any possible explanations for this distinction?

More on error analysis...

(11) Using $P_{14}(x)$ and x = 6, what will the error be?

(12) Again using $P_{14}(x)$, what are the possibilities for x if the error is to be less than 0.0005?

(13) If x = 4, how many terms are necessary to be sure that the error is less than .01?

Taylor Series 3

It's time to generalize the process. Assume we have an arbitrary function f, and assume that at a = 0, $f(0)$ and the derivatives $f'(0)$, $f''(0)$, ... , and $f^{(n)}(0)$ all exist.

Let $p_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$ be an n^{th} degree polynomial.

If we set $p_n^{(k)}(0) = f^{(k)}(0)$ for k = 0, 1, 2, ..., n, we can solve for a_k, still for k = 0, 1, ..., n.

(1) Set $f(0) = p_n(0)$ to find a_0 in terms of $f(0)$.

(2) Set $f'(0) = p_n'(0)$ to find a_1 in terms of $f'(0)$.

(3) Set $f''(0) = p_n''(0)$ to find a_2.

(4) Set $f'''(0) = p_n'''(0)$ to find a_3.

(5) Find $p_n^{(n)}(x)$ and set $f^{(n)}(0) = p_n^{(n)}(0)$ to find a_n in terms of $f^{(n)}(0)$.

(6) Write the polynomial $p_n(x)$ in terms of the derivatives of f evaluated at 0.
(This is called either the n^{th} order Taylor polynomial at a = 0 **or** the n^{th} order Maclaurin polynomial for the function f.)

Series 4

Popular Maclaurin Series
and their intervals of convergence!

Write the first four terms of the following series and state the interval of convergence. Try do do these without looking them up.

Interval

e^x =

sin x =

cos x =

$\mathbf{\frac{1}{1-x}}$ =

(1) Use substitution to find series for each of the following.

(a) sin 3x =

(b) e^{4x} =

(c) cos (x/2) =

(d) $\frac{1}{e^{2x}}$ =

(e) e^{x^2} =

Please note that the interval of convergence for all of these is still .

(2) Write the series for $\frac{1}{1-2x}$. Find the interval of convergence, again using substitution.

Write the series for $\frac{1}{1+3x}$. Find the interval of convergence.

Write the series for $\frac{3}{3-2x}$ by writing this as $\frac{1}{1-2x/3}$. Find the interval of convergence.

(3) Logarithms. (You may need to look this up or derive this.)

<u>Series</u> <u>Interval of convergence</u>

ln (1 + x) =

ln (1 + 5x) =

ln (1 - x/3) =

Series 5

New from Old

Complete the right side of the equations below with the appropriate series and the interval of convergence if requested.

(1) sin x =

Differentiate both sides.

Does this "work"?

Differentiate both sides again. Does this still "work"?

(2) e^x =

Integrate both sides.

Does this "work"? What do you <u>always</u> remember to include when integrating?

(3) <u>Interval of convergence</u>

$\frac{1}{1-x}$ =

$\frac{1}{1+x}$ =

ln (1 + x) =

Note the endpoints, particularly on the integral.

(4) Interval of convergence

$\frac{1}{1-x}$ =

$\frac{1}{1+x^2}$ =

$\tan^{-1}x$ =

Note that this is the second method we've seen for finding the Maclaurin series for $\tan^{-1}x$. Any preference?

Theorem: (Simplified) Taylor series may be differentiated or integrated term by term and will represent the appropriate function on the same open interval.
Note: Convergence or divergence may change at each endpoint.

(5) Find, using simple multiplication.

x^2e^x =

$4x\cos(2x)$ =

(6) Multiplication of series. Find the series for $\frac{e^x}{1 - x}$ by multiplying enough terms of each of the two series together. (Be careful. Think ahead.)

(7) Let $f(x) = \dfrac{5x - 1}{x^2 - x - 2}$. Finding the derivatives of this to create the series directly would be somewhat unpleasant. Try this approach:

(a) Use partial fractions to decompose f.

(b) Write the series for each of the two partial fractions. (One may require a bit of sneakiness.)

(c) Add and simplify.

(8) Let's try that last function by an additional method: Long division. Find three terms.

$$-2 - x + x^2 \,\overline{\big)\, -1 + 5x \quad}$$

Series

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In no order whatsoever...

(1) Use series to evaluate the limit: $\lim_{x \to 0} \frac{1-\cos(3x)}{x^2}$

Use l'Hôpital's rule to check your work below: $\lim_{x \to 0} \frac{1-\cos(3x)}{x^2}$

(2) If a Taylor series for a function f converges on the interval [-1, 5), find the interval of convergence for f(2x).

find the interval of convergence for f(-x).

(3) The Maclaurin series for a function is given below.
$\frac{x-1}{x^2-2x-3} = \frac{1}{3} - \frac{5x}{9} + \frac{10x^2}{27} - \frac{41x^3}{81} + \ldots$

Use this to find the first four terms of the series for the following:
$\frac{1}{2}\ln\left(x^2-2x-3\right) =$

(4) Write the first four terms of the series for each of the following functions.
sin(x/2), $\frac{2}{e^x}$, $\frac{x}{1-2x}$

(5) Give an example of a series which converges on each of the following intervals.
[-3, 3]   [-1, 4]

(6) Given $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2+5}$,

Estimate the error if 20 terms are used.

Find S_{20} and then find an interval for S. (That is, find upper and lower bounds for S using S_{20}.)

How many terms are necessary to be sure that the error is less than .01?

(7) If there is a Maclaurin series for the function f, and $f^{(n)}(x) \le n + 2$ for all x, find an upper bound for the error using terms through n = 6 of the series when approximating $f(1.5)$.

(8) If 4 non-zero terms are used to approximate cos2, find an upper bound for the error.

(9) If 3 non-zero terms are used to approximate sinx with an error less than .001, what values of x may be used?

(10) To approximate $e^{.7}$ with 5 terms (through n = 4), find an upper bound for the error.

(11) The Maclaurin series for some function f is

$$f(x) = \frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots + \frac{x^n}{(n+2)!} + \ldots$$

Find $f'(0)$. Find $f^{(8)}(0)$.

(12) State (only) whether each series is absolutely convergent, conditionally convergent or divergent.

$\sum \frac{(-1)^{k+1}k}{k^3+1}$ $\sum \frac{(-1)^k}{k \ln k}$ $\sum (-1)^n (n\sin(1/n))$

$\sum \frac{(-2)^k}{k!}$ $\sum (-1)^{m+1} \sqrt[m]{m+1}$

(13) Find the interval of convergence. (Show work carefully.)

$$\sum_{k=1}^{\infty} \frac{(x+3)^k}{k\ 4^k}$$

(14) Give an example of a series which converges on [2, 10].

(15) Given $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2+3}$. (Show set-up clearly, particularly for calculator work.)

(a) Find an upper bound for the error if 8 terms are used.
(b) Find an inequality which bounds S.
(c) How many terms must be used to be sure the error is less than 0.005?

(16) Give an example of an alternating series...

which is conditionally convergent.
which is divergent.
which is absolutely convergent.

(17) For the series $\sum_{\boldsymbol{n}=1}^{\infty} \boldsymbol{x_n}$ the partial sums are $S_n = \frac{3\boldsymbol{n}}{\boldsymbol{n}+2}$. Find x_4.

Find the sum of the series, if it converges.

(18) If terms through n = 10 are used to approximate $e^{2.8}$, find an upper bound for the error.

(19) Find the Maclaurin series for each of the following. Show 4 terms.

$\sin(x^2) =$ $\qquad$ $\frac{x^2}{e^x} =$ $\qquad$ $\frac{3}{1+2x} =$

(20) Approximate $\int_0^{0.5} \frac{dx}{1+x^3}$ with an error less than .001.

(21) The Maclaurin series for some function k is

$$k(x) = \frac{x}{2!} + \frac{2x^2}{3!} + \frac{3x^3}{4!} + \ldots + \frac{nx^n}{(n+1)!} + \ldots$$

Find k'(0). $\qquad$ Find $k^{(12)}(0)$.

(22) Determine the values of x for which the series $\sum_{k=0}^{\infty} 5(3+x)^k$ converges. Then find S in terms of x. (Note: This problem is meant to be given while studying geometric series, before the ratio test is introduced.)

(23) Can the ratio test be used on $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^2}$? Why or why not?

Can the limit comparison test be used on $\sum_{k=1}^{\infty} \frac{2+\sin k}{k^3}$? Why or why not?

(24) Estimate $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+k}}$ on your calculator. (Indicate your method.) Do you think this is a good estimate? Why or why not? How could you check this?

AP problems – all BC
1975:4, 1976:7, 1977:5, 1978:5, 1979:4, 1980:3, 1981:3, 1982:5, 1983:5, 1984:4, 1986:5, 1987:4, 1988:4, 1990:5, 1991:5, 1992:6, 1993:5, 1994:5, 1995:4, 1996:2, 1997:2, 1998:3, 1999:4, 2000:3

1987:4

(a) Find the first five terms in the Taylor series about x = 0 for $f(x) = \frac{1}{1-2x}$.

(b) Find the interval of convergence for the series in part (a).

(c) Use partial fractions and the result from part (a) to find the first five terms in the Taylor series about x = 0 for $g(x) = \frac{1}{(1-2x)(1-x)}$.

1993:5

Let *f* be the function given by *f*(x) = $e^{x/2}$.

(a) Write the first four nonzero terms and the general term for the Taylor series expansion of *f*(x) about x = 0.

(b) Use the result from part (a) to write the first three nonzero terms and the general term of the series expansion about x = 0 for $g(x) = \frac{e^{x/2}-1}{x}$.

(c) For the function g in part (b), find g'(2) and use it to show that $\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$.