Ages 17-19 – You must still think yourself, CAS can only help

a) A classroom activity and instructions for teachers

(i) Ask the students to draw the graph of a cubic, e.g. $f(x) = x^3 - 5 \cdot x^2 + 2 \cdot x + 4$. They must all have the same graph.

(ii) Each student now chooses a linear function $g(x) = p \cdot x + q$, such that the line intersects the cubic three times. Have each student choose a different linear function.

(iii) Determine the solutions $x1$, $x2$, and $x3$ to the equation $f(x) = g(x)$ and calculate the sum of the solutions, $x1 + x2 + x3$. The values of the three solutions can be found using either the graph or solve.

The most direct way to obtain the sum is to write: `sum(zeros(f(x)-g(x),x))`.

It will definitely surprise the students that even though they have different values for the solutions $x1$, $x2$, and $x3$, they will all obtain the same `sum(x1+x2+x3=5)`.

(iv) Repeat the above procedure with another cubic.

(v) Conjecture

For any cubic $f(x) = x^3 + a \cdot x^2 + b \cdot x + c$ the graph of which intersects a line at three points, the sum of the x-values of the intersection points is $-a$ regardless of which line is drawn.

b) Solution for (v)

Proving the conjecture cannot be done directly with a Voyage 200. If you try to solve the equation $x^3 + a \cdot x^2 + b \cdot x + c = p \cdot x + q$ (*), you won't get anywhere.

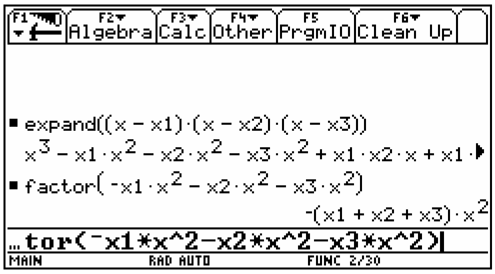
You simply do not get an expression for the solutions. The proof is easy – also by hand – but you must have an approach for solving the problem.

Proof

The solutions $x1$, $x2$ and $x3$ to equation (*) are the zeros of the polynomial $h(x)$ where:

$h(x) = f(x) - g(x) = x^3 + a \cdot x^2 + (b - p) \cdot x + c - q$. This means that

$h(x) = (x - x1) \cdot (x - x2) \cdot (x - x3)$, and at this point many students will choose to use their Voyage 200.

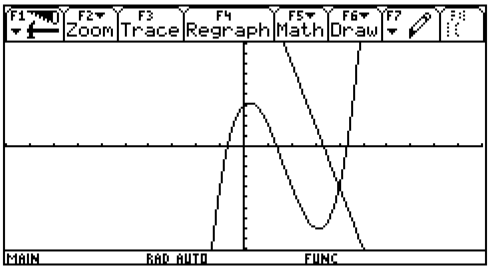


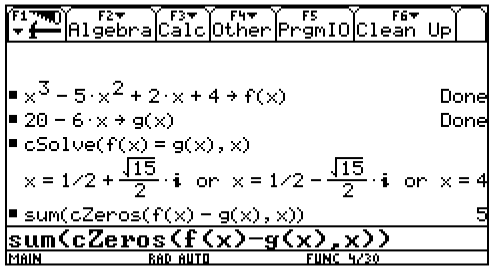
This concludes the proof. The basic reason why we get the same sum for all lines is that the values of p and q do not affect the coefficient of the term of degree 2.

This simple example demonstrates the importance of good mathematical understanding for solving problems in a CAS classroom. Students will not be able to use CAS to solve this problem unless they have good mathematical understanding of the problem.

c) (i) Example

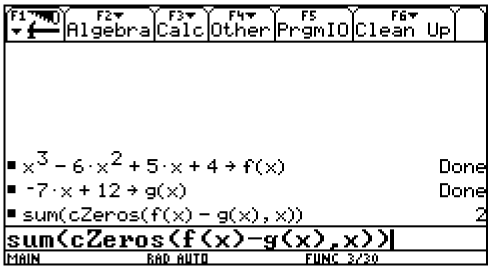
With $f(x) = x^3 - 5 \cdot x^2 + 2 \cdot x + 4$ and $g(x) = -6 \cdot x + 20$ there is only one point of intersection. The equation $f(x) = g(x)$ has one real solution and two imaginary solutions. The proof in b) can be extended to complex numbers without any changes.





(ii) Exercise

With $f(x) = x^3 - 6 \cdot x^2 + 5 \cdot x + 4$ and $g(x) = -7 \cdot x + 12$ we get



What is wrong? Why is the sum of zeros not equal to 6 ?

(iii) Exercise

Study $f(x) = x^3 - 6 \cdot x^2 + 5 \cdot x + 4$ and $g(x) = -4 \cdot x + 8$ the same way as above, and try to formulate a general theorem about the sum of solutions for any equation $x^3 + a \cdot x^2 + b \cdot x + c = p \cdot x + q$.

d) Exercise

Draw the graph of the quartic $f(x) = x^4 - 2 \cdot x^3 - 3 \cdot x^2 + 5 \cdot x - 3$ and choose a linear function $g(x) = p \cdot x + q$ to give four intersection points. Answer questions (1)-(3) given above for this quartic.