

NUMB3RS Activity: Coded Messages Episode: "The Mole"

Topic: Inverse Matrices

Grade Level: 10 - 11

Objective: Students will learn how to apply inverse matrix multiplication to the coding of values.

Time: 15 minutes

Materials: TI-83 Plus/TI-84 Plus graphing calculator

Introduction

In "The Mole," the FBI discovers that a suspect downloaded a number of images off the Internet. Charlie points out that the NSA has discovered groups like al-Qaeda and Hezbollah hide messages in photographs through a process called steganography. On a computer screen, a picture is made up of a series of thousands of dots and bytes, and not every single byte is needed to make the picture. A coded message can easily be hidden in the unused parts of the image.

Discuss with Students

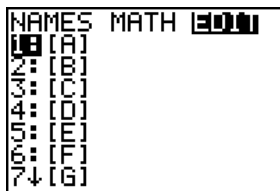
- For the main activity, students should already be familiar with matrix multiplication. An example is given below, in case a review is needed.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 9 & 12 \\ 0 & 10 & 13 \\ 8 & 11 & 14 \end{bmatrix} = \begin{bmatrix} 1(7) + 2(0) + 3(8) & 1(9) + 2(10) + 3(11) & 1(12) + 2(13) + 3(14) \\ 4(7) + 5(0) + 6(8) & 4(9) + 5(10) + 6(11) & 4(12) + 5(13) + 6(14) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 62 & 80 \\ 76 & 152 & 167 \end{bmatrix}$$

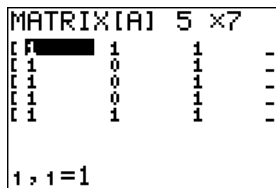
- Remind students of the following matrix properties.
 - Matrix multiplication is not commutative (i.e., AB is not necessarily equal to BA)
 - To multiply matrices A and B , the number of columns in A must equal the number of rows in B . To illustrate this, reverse the multiplication with the above matrices.
- In the activity, students will use their calculators to do most of the computations. The guide below gives a quick overview of how to manipulate matrices on the calculator.

1.



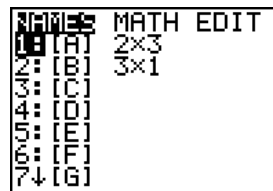
To enter a matrix A, press 2nd [MATRIX], go to the **EDIT** menu, and select **1:[A]**.

2.



Enter the number of rows and columns. Then input the entries of the matrix.

3.



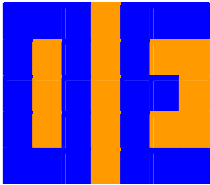
To multiply two matrices from the home screen, press 2nd [MATRIX] and from the **NAMES** menu, select the appropriate matrices.

- Students may not be aware that some matrices do not have inverse. In order for a matrix to have an inverse, its determinant cannot equal 0. For example, the matrix $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ does not have an inverse, because $\det \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} = 3(4) - 6(2) = 0$.

Student Page Answers:

1. $\begin{bmatrix} 9 & 9 \\ 23 & 15 \end{bmatrix}$ 2. $-\frac{1}{9} \begin{bmatrix} 0 & -3 \\ -3 & 1 \end{bmatrix}$ 3. The identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. 4. The matrix itself. This is the matrix version of the identity property of multiplication.

5.



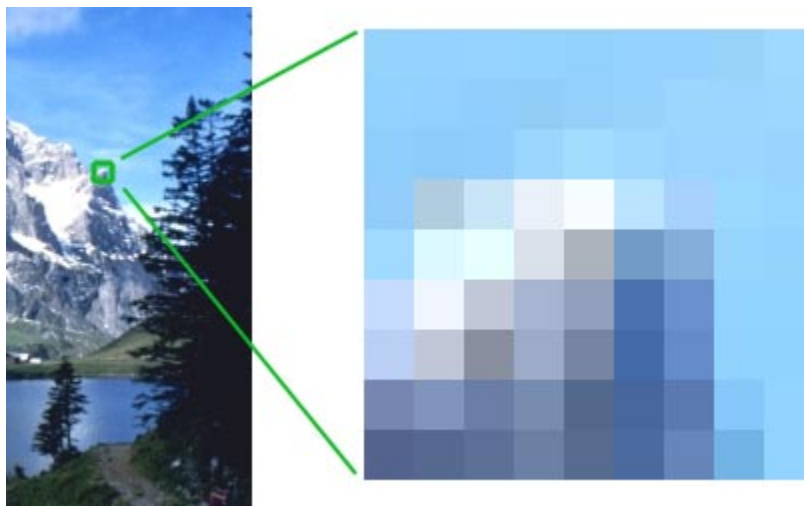
Name: _____

Date: _____

NUMB3RS Activity: Coded Messages

In "The Mole," the FBI discovers that a suspect downloaded a number of images off the Internet. Charlie points out that the NSA has discovered groups like al-Qaeda and Hezbollah hide messages in photographs through a process called steganography. On a computer screen, a picture is made up of a series of thousands of dots and bytes, and not every single byte is needed to make the picture. A coded message can easily be hidden in the unused parts of the image.

How does this work? Every bitmap picture viewed on a computer is made up of thousands of very small squares called pixels, as shown in the picture below.



[Source: <http://www.prepressure.com/image/formatbitmap.jpg>]

The computer views these pixels as numbers, not colors. For example, the pixelated image above could be translated into the values below:

$$\begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 14 & 8 & 7 & 8 & 14 & 12 & \dots \\ 7 & 6 & 12 & 8 & 6 & 7 & \dots \\ 4 & 4 & 4 & 3 & 12 & 14 & \dots \end{bmatrix}$$

Someone sending a secret message as a picture could use encryption through matrix multiplication.

- To encrypt matrix $A = \begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix}$ with the encryption matrix $E = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$, find AE .

The obtained result is the message sent to the receiver. The receiver now needs to decode and read the message. This decrypting involves the inverse matrix.

For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ its inverse is calculated $\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

2. Calculate the inverse of the encryption matrix from Question 1, $\begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$.

3. Multiply this inverse matrix by the original encryption matrix. What is the result?

This answer to Question 3 is called the identity matrix. This matrix is the key to finding the inverse of a matrix.

4. Multiply any 2×2 matrix by the answer from Question 3. What is the result?

5. There are many ways to calculate the inverse of a matrix larger than 2×2 . For this activity, you will use your calculator to find the inverse.

Suppose the following message and encryption key were just intercepted. Using your calculator, multiply the inverse of the encryption key by the intercepted message and decode the picture using the color values below.

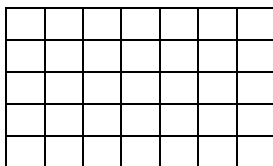
0 Black	1 Red-Brown	2 Green	3 Brown-Green
4 Dark Blue	5 Purple	6 Blue-Green	7 Dark Gray
8 Light Gray	9 Orange-Red	10 Vivid Green	11 Yellow
12 Blue	13 Pink-Purple	14 Light Blue	15 White

$$\begin{bmatrix} 29 & 28 & 32 & 37 & 25 & 29 & 20 \\ 44 & 58 & 47 & 62 & 45 & 44 & 25 \\ 34 & 38 & 42 & 47 & 35 & 39 & 25 \\ 44 & 58 & 47 & 62 & 45 & 44 & 25 \\ 29 & 28 & 32 & 37 & 25 & 29 & 20 \end{bmatrix}$$

Intercepted Message

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 1 & 2 \\ 2 & 4 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 2 & 0 \end{bmatrix}$$

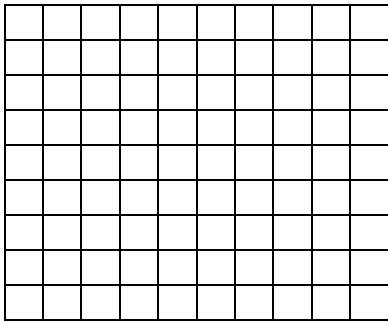
Encryption key



Decoded Picture

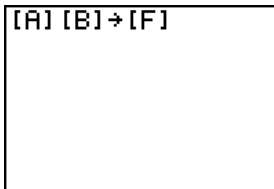
Now, reverse roles – encrypt your own message and send it to another graphing calculator.

Color the grid below with a picture to encrypt. Use the color codes from the previous page to store the image as matrix *A*. Store the encryption matrix at the right below as matrix *B*.



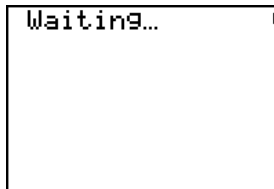
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

1.



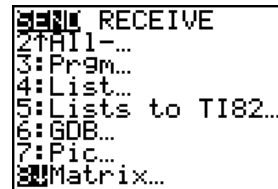
To encode the picture matrix *A* with the encryption matrix *B*, find *AB*. Store the result as a matrix different from your neighbor by pressing the **STO** key and selecting a matrix from the matrix menu.

2.



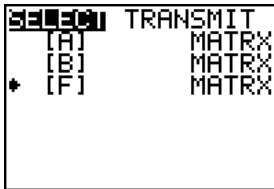
To send this coded message to your neighbor's calculator, make sure both calculators are connected with the unit-to-unit cable. On the calculator receiving the matrix, press **2nd** [LINK], go to the **RECEIVE** menu, and press **ENTER**.

3.



On the calculator sending the matrix, press **2nd** [LINK], go to the **SEND** menu, and select **8:Matrix...**

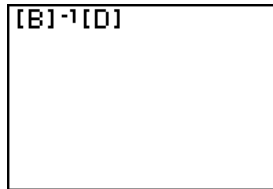
4.



Select the coded matrix, press **ENTER**, and select **TRANSMIT** to send.

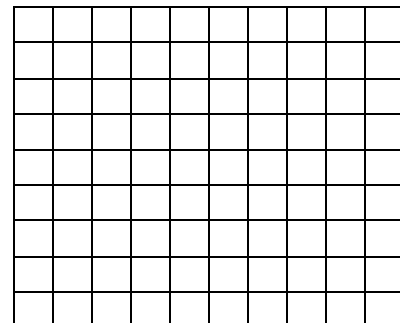
You neighbor will now send an encrypted matrix to you.

5.



To decode the encoded matrix, multiply the inverse of the encryption matrix by the encoded matrix.

Using the decoded matrix, color the squares to the right based on the color coding used earlier. Check with your neighbor if you decoded it correctly.



The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research

Extensions

Activity: Finding the Inverse Matrix by the Adjoint Method

Introduction

Two common methods for finding the inverse of a matrix are Gauss-Jordan and the Adjoint method. The Adjoint method is described below. To find A^{-1} , calculate

$$A^{-1} = \frac{1}{\det A}(\text{adj } A), \text{ where adj } A \text{ is the transpose of the cofactor matrix of } A.$$

Example: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 0 & 3 \end{bmatrix}$ find A^{-1} .

Step 1: Find the cofactor matrix of A . This is found by replacing each cell in matrix A with its minor and imposing an alternating negative sign.

$$\begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 9 & 3 & -9 \\ -6 & -6 & 6 \\ -5 & 1 & 1 \end{bmatrix}$$

Step 2: Find the transpose of the cofactor matrix. To do this, interchange the rows and columns of the matrix. This means make the columns into rows.

$$\begin{bmatrix} 9 & -6 & -5 \\ 3 & -6 & 1 \\ -9 & 6 & 1 \end{bmatrix}$$

Step 3: To find A^{-1} , multiply the transposed matrix by $\frac{1}{\det A}$

$$-\frac{1}{12} \begin{bmatrix} 9 & -6 & -5 \\ 3 & -6 & 1 \\ -9 & 6 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} & \frac{5}{12} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{12} \\ \frac{3}{4} & -\frac{1}{2} & -\frac{1}{12} \end{bmatrix}$$

Additional Resources

- The actual images used in the *NUMB3RS* episode were JPEG images (not bitmaps). The different methods of steganography for JPEG images can be found at the Web site <http://www.guillermi2.net/stegano/jsteg/index.html>.
- To learn a different technique of using steganography to disguise a message, see the *NUMB3RS* activity "Now You See It." This activity can be downloaded for free by going to <http://education.ti.com/exchange> and searching for "6213."
- To learn more on multiplying matrices, see the *NUMB3RS* activity "Settling the Score." This activity can be downloaded for free by going to <http://education.ti.com/exchange> and searching for "6809."