

# EXPLORATIONS

## Chapter 2

### Differentiation

In this chapter, you will explore estimated and exact results for derivatives, tangent lines, implicit differentiation, and symbolic differentiation.

#### Example 1: Finding slope and the tangent line

In this example, you will use **avgRC**, the average rate of change command, and **nDeriv**, the symmetric difference quotient. These commands introduce difference quotients and how they are used to estimate slope. The limits of these difference quotients then define the derivative.

Given

$$f(x) = \sqrt{x^2 + 1},$$

estimate the slope of the curve when  $x=1$ . Determine the equation of the tangent line.

#### Solution

Use the traditional difference quotient (average rate of change) and the symmetric difference quotient (numerical derivative) to estimate the slope. Also, use a tangent command to draw the line and display the equation.

#### Calculating average rate of change

1. If needed, press **[HOME]** to return to the Home screen. You will now define a function. This definition will remain in the calculator's memory until it is cleared. Press **[F4]** **Other** and select **1:Define**.
2. Define the function.

**F** **[ ]** **X** **[ ]** **=** **[2nd]** **[√]** **X** **[^]** **2** **[+]** **1** **[ ]** **ENTER**



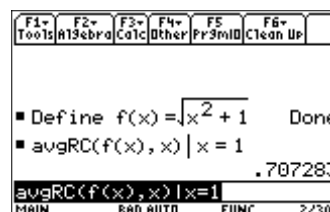
3. Press **CATALOG**. If needed, press **A** to move to commands that begin with A. Press **2nd**  $\odot$  to “page down” to **avgRC**. Press **ENTER**.



4. Complete the command.

**F**  $\left[ \left[ X \right] \left[ , \right] \left[ X \right] \left[ ] \right] \left[ X \right] \left[ = \right] \left[ 1 \right] \left[ \text{ENTER} \right]$ .

Note that the default increment for the average rate of change is .001. Thus, the slope of the line through the points with  $x$ -coordinates 1 and 1.001 has been computed and is displayed. It is an estimate of the slope of the tangent line, or the slope of the curve, at the point  $x=1$ .



5. You can use a different increment if you enter it as the fourth argument of the **avgRC** command. To use .0001, for example, press  $\odot$ . The cursor is now at the right of the expression. Press  $\odot$  five times to move between the second  $x$  and the right parenthesis. Press  $\left[ , \right]$ , type **.0001** and press **ENTER**.

This time the slope of the line through the points with  $x$ -coordinates 1 and 1.0001 has been computed and displayed. It may be a better estimate of the slope of the tangent line. Also, note that the increment value can be entered as a negative number, so that the slope line may be computed using a point to the left of the given point.



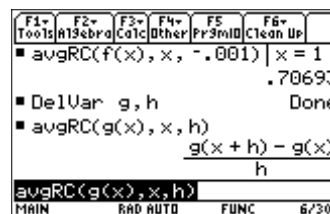
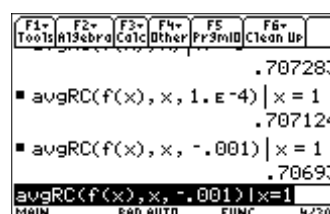
6. To gain some insight into the **avgRC** command, you can use an undefined function such as  $g(x)$ . You want to be sure that it is not currently defined; however, you may not want to clear other variables such as the function  $f$  that was just defined.

The best way to clear a few specified variables is by a **DelVar** command. To guarantee that  $g$  and  $h$  are undefined, press **F4 Other**, select **4:DelVar**, enter **G**  $\left[ , \right]$  **H** and press **ENTER**.

Now press **CATALOG** to select **avgRC**. The selection arrow will still be at **avgRC** if you haven't done anything else in the CATALOG since step 3. Complete the command.

**G**  $\left[ \left[ X \right] \left[ , \right] \left[ X \right] \left[ , \right] \left[ H \right] \left[ ] \right] \left[ \text{ENTER} \right]$

You now have an average rate of change for an arbitrary function with an arbitrary increment.



### Finding the numerical derivative

If you compute an average rate of change for one positive and one negative value for  $h$ , you will often get one value that is an overestimate and one value that is an underestimate. You can average these results for another estimate. The same result can be obtained directly using the symmetric difference quotient, often referred to as the *numerical derivative*. It has been used more often recently since graphing calculators began using it for numerical differentiation estimates.

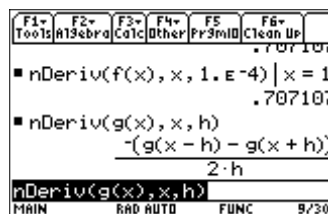
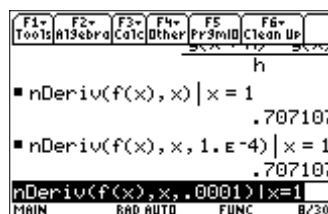
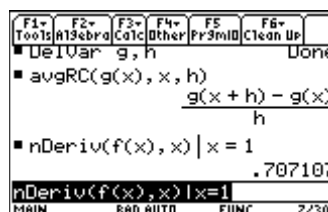
- To try the same problem with a symmetric difference quotient, press **F3** **Calc** and select **A:nDeriv(**. Complete the command.

**F** (**□** **X** **□** **,** **X** **□** **□** **X** **□** **-** **1** **□** **ENTER**).

Again, the default increment is .001.

- To try a smaller increment, insert **.0001** as the third argument, just as you did for **avgRC(**, and press **ENTER**.

- To gain some insight into the **nDeriv(** command, use the command on  $g(x)$  with increment  $h$  as you did for **avgRC(**.

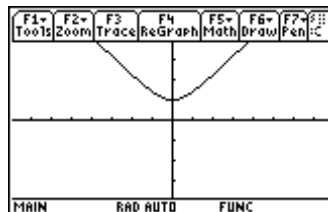
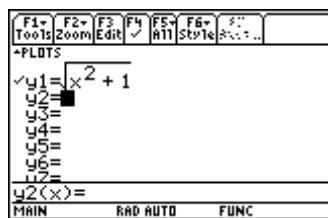


### Drawing the tangent line

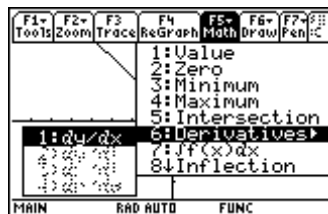
- To enter the function for graphing, press **♦** [**Y=**]. If necessary, press **⊖** or **⊕** to move the cursor to  $y1$ . If there is an old equation, press **CLEAR**. As you type the function, it appears on an entry line near the bottom of the screen.

**2nd** [**√**] **X** **^** **2** **+** **1** **□** **ENTER**.

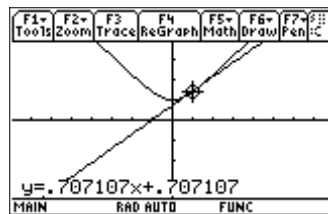
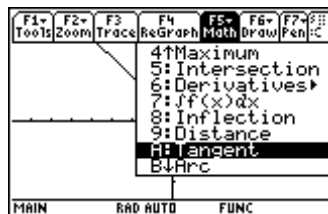
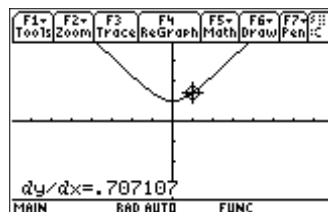
- To set a convenient viewing window and center the origin, press **F2** **Zoom** and select **4:ZoomDec**.



3. Compute the slope at a point on the graph. First, press **[F5] Math** and select **6:Derivatives**, and then select **1:dy/dx**. Now you are prompted for the  $x$ -coordinate. You can use the arrow keys, but it is easier just to type a value. For example, to input 1 as the  $x$ -coordinate, press **1 [ENTER]**. The estimated value of the derivative is displayed.



4. To plot a tangent line and display the equation, press **[F5] Math** and select **A:Tangent**. Enter 1 for the  $x$ -coordinate. The tangent line and its equation are displayed.



### Example 2: Exact derivatives

Each of the methods so far have used an estimation method for the slope. With the TI-89, you can compute exact derivatives as well as the exact value for the slope at a given point.

Given the same function,

$$f(x) = \sqrt{x^2 + 1},$$

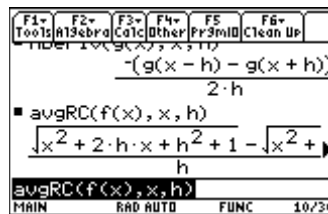
find  $f'(x)$ ,  $f''(x)$ , and the exact value of  $f'(1)$ .

**Solution**

Explore the derivatives with limits and the built-in derivative command.

- To explore with the traditional difference quotient, press **CATALOG**, select **avgRC**(, and press **ENTER**. Complete the command to display the traditional difference quotient with increment  $h$  for this function.

**F** (**X**) **,** **X** **,** **H** **)** **ENTER**



- Of course, the derivative is the limit as  $h$  approaches zero of the expression you have just computed. To evaluate this limit, press **2nd** to move the cursor to the beginning of the command. Then press **F3** **Calc** and select **3:limit**(. This will paste the **limit**( command at the beginning of the command.



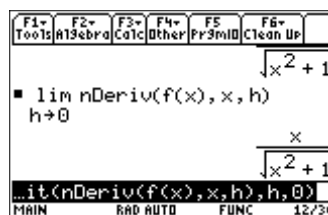
- Now finish the command with the limit variable and value by pressing **2nd** to move the cursor to the end of the command line. Press **H** **,** **0** **)** **ENTER**. The actual derivative is displayed, and it has been computed by the definition of the derivative.



- You also can compute a derivative as a limit of the symmetric difference quotient. Enter the command:

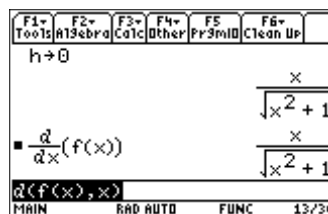
**F3** **3:limit**( **F3** **A:nDeriv**( **F** (**X**) **,** **X** **,** **H** **)** **,** **H** **,** **0** **)** **ENTER**

The actual derivative is displayed, and it has been computed as a limit of the symmetric difference quotient.

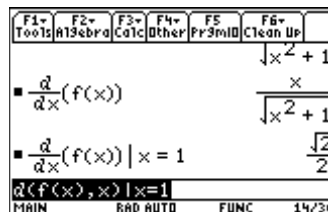


- Although the limits computed above are important for the concept of the derivative, they are not the easiest way to compute the derivative. Use the built-in derivative command for the function  $f(x)$ .

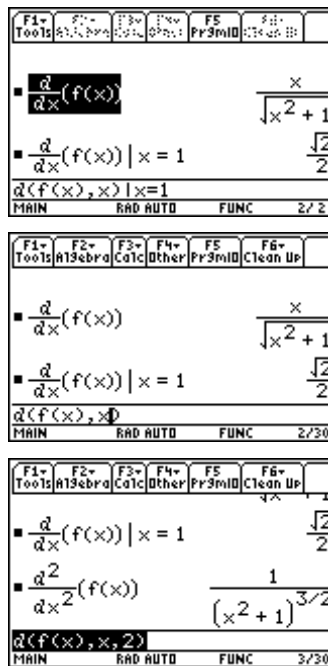
**2nd** **[d]** **F** (**X**) **,** **X** **)** **ENTER**



- You also can evaluate the derivative at a particular point. The command uses the “with” operator that you used in Example 1 to estimate the slope at  $x=1$ . To compute  $f'(1)$ , press **2nd** so the cursor is at the end of the entry line of your previous command. Then press **X** **=** **1** **ENTER**. The exact slope at  $x=1$  is displayed.



7. You can compute higher order derivatives by typing a value as the third argument of a differentiate command. To compute the second derivative, use the original derivative command (without  $x=1$ ). You can move through the history area to recall that command. Move the insert cursor between the second  $x$  and the right parenthesis. Press  $\boxed{,} \boxed{2} \boxed{\text{ENTER}}$ .



### Example 3: Implicit differentiation

You can compute derivatives for implicit functions in several ways.

Given  $x^2 + xy - y^2 + 2x - 3y - 12$ , compute  $\frac{dy}{dx}$  and evaluate it when  $x=4$ .

#### Solution

Use Implicit Function Theorem from multivariable calculus to compute the derivative in terms of  $x$  and  $y$ . Solve for the  $y$ -coordinates when  $x=4$  and substitute. Alternately, enter  $y$  as an (unknown) function of  $x$  and compute the derivative directly. Finally, compute  $y$  as an explicit function of  $x$ , compute the derivative in terms of  $x$  only, and substitute.

#### Using multivariable calculus

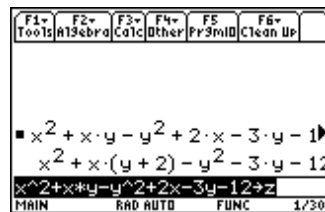
Implicit Function Theorem states that if  $z = f(x, y)$ ,

$$\text{then } \frac{dy}{dx} = \frac{-\partial z / \partial x}{\partial z / \partial y}.$$

Consult a multivariable calculus text for more details on this theorem.

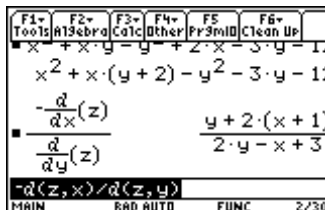
1. First, store the expression to  $z$ .

$X \wedge 2 + X \times Y - Y \wedge 2 + 2X - 3Y - 12$  [STO] [Z] [ENTER]



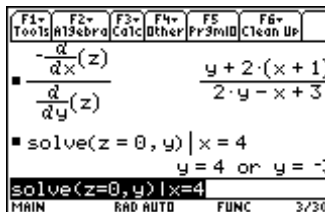
2. Apply Implicit Function Theorem and compute the result for implicit differentiation.

[(-)] [2nd] [d] [Z] [X] [÷] [2nd] [d] [Z] [Y] [ENTER]



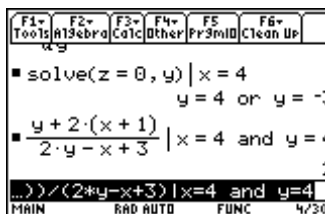
3. Solve for the  $y$ -coordinates when  $x=4$ .

[F2] 1:solve( [Z] = 0 [Y] [ ] [X] = 4 [ENTER]

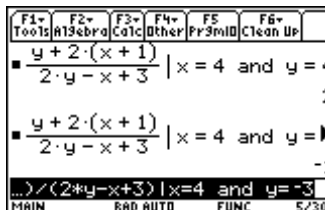


4. To compute the slope at (4,4), highlight the slope formula in the history area and press [ENTER]. Use the “with” operator and substitute for  $x$  and  $y$  to compute and display the slope (4,4).

[ ] [X] = 4 [CATALOG] and [Y] = 4 [ENTER]



5. To compute the slope at the other point, press [ ] [←] [(-)] 3 [ENTER] to change the  $y$ -coordinate to -3.

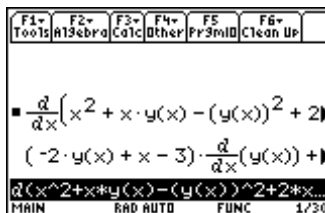


**Computing the derivative directly**

You may wish to compute the derivative without use of a multivariable calculus theorem. Two important details are: 1) it is necessary to write  $y$  as a function of  $x$ , rather than just the variable  $y$  and 2) some creativity is needed to get  $dy/dx$  alone for the final result.

1. Enter the differentiate command for  $d(x^2 + x*y(x) - (y(x))^2 + 2x - 3y(x) - 12, x)$ .

[2nd] [d] [X] [^] 2 [+] [X] [X] [Y] [ ( ] [X] [ ) ] [-] [ ( ] [Y] [ ( ] [X] [ ) ] [ ) ] [^] 2 [+] 2 [X] [-] 3 [Y] [ ( ] [X] [ ) ] [-] 12 [ , ] [X] [ ) ] [ENTER]



2. You now have a solution for the derivative. However, usually the solution is rearranged with  $\frac{dy}{dx}$  isolated.

The **solve**( command on the TI-89 will not accept the derivative as the variable to solve for, so you must substitute first. For instance, you might use  $dydx$  (the four letters, not a built-in symbol).

Press  $\leftarrow$  [ENTER] to paste the previous to the entry line, and complete the expression.

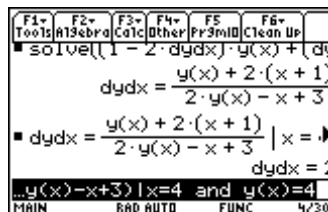
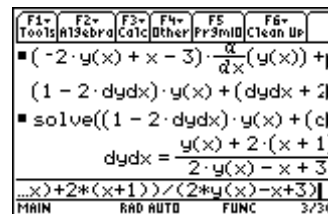
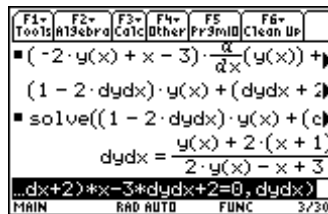
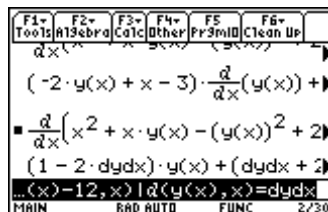
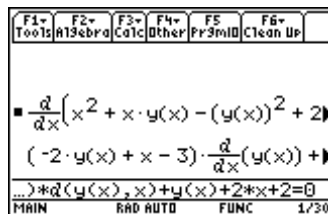
[1] [2nd] [d] [Y] ([X]) [ , ] [X] [ ] [=] **DYDX** [ENTER]

3. To solve, press [CLEAR] to clear the entry line. Then enter the command:

[F2] **1:solve**(  $\leftarrow$  [ENTER] [=] 0 [ , ] **DYDX** [ ] [ENTER]

4. To evaluate the derivative at the point (4,4), edit the previous result.

[CLEAR]  $\leftarrow$  [ENTER] [1] [X] [=] 4 [CATALOG] and [Y] ([X]) [=] 4 [ENTER]

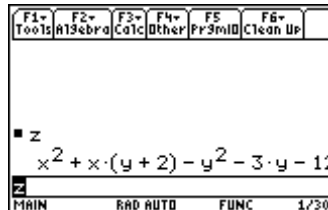




### Solving for the function $y(x)$

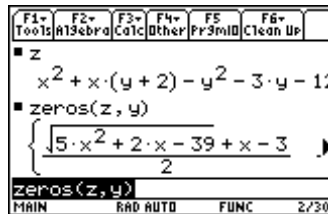
In some, but not all implicit differentiation problems it is possible to find  $y$  as an explicit function of  $x$ . In this example, you can see that the relation is quadratic in  $y$  and you can solve for  $y$ .

1. To solve for  $y$  in terms of  $x$ , first check that  $z$  is still defined. You can do this by pressing **Z** **ENTER**.

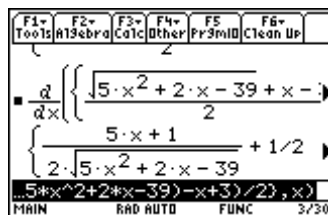


2. Use **zeros**( to find the values of  $y$  that make  $z$  equal to zero.

**F2** 4:**zeros**( **Z** , **Y** ) **ENTER**

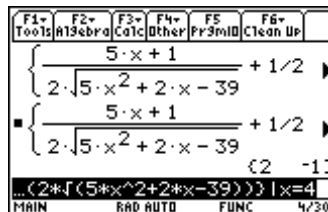


3. To differentiate, press **2nd** [**d**] and then **ENTER** to paste the previous result. Now press **X** **ENTER**.

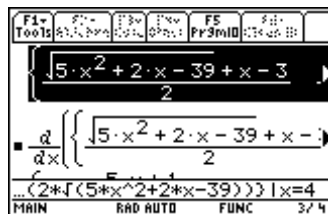


4. Evaluate the derivative when  $x=4$ .

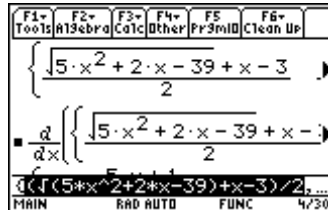
**CLEAR** **ENTER** **I** **X** = **4** **ENTER**



5. You also can plot the implicit function and compute the value of the derivative at  $x=4$  from the graph screen. For this approach, you can paste the result you have from the **zeros**( command to the **Y=** Editor. Press **CLEAR** and then press **ENTER** five times to highlight the result of the **zeros**( command.



Press **ENTER** to paste to the command line, **2nd** **ENTER** to move the cursor to the beginning of the entry line, and **ENTER** once to move the cursor between the { and the (.



Now press and hold  $\uparrow$  while pressing  $\rightarrow$  to highlight the first expression all the way to, but not including, the comma. Press  $\boxed{F1}$  **Tools** and select **5:Copy** to copy the expression.

- Press  $\boxed{\diamond}$   $[Y=]$ , check that the cursor is on  $y1$ , clear any function you may have in  $y1$ . Press  $\boxed{F1}$  **Tools**, and select **6:Paste**. The expression is pasted to the entry line. Press  $\boxed{ENTER}$  to paste the expression in  $y1$ .

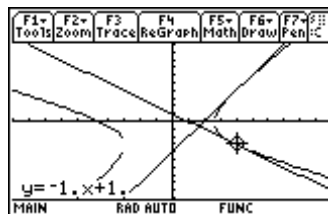
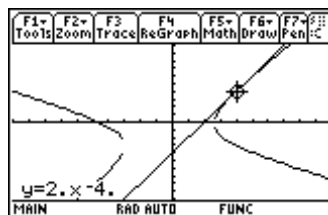
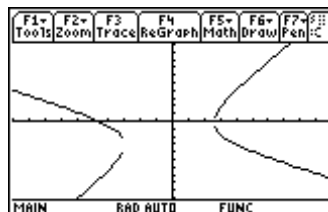
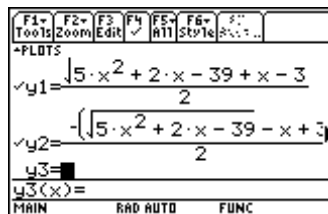
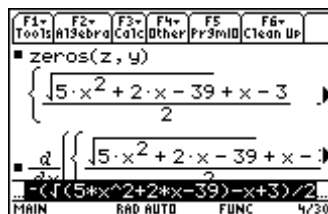
- Press  $\boxed{HOME}$ . The previous command should still be on the entry line. Press  $\rightarrow$  to move to the right of the comma. Press and hold  $\uparrow$  while pressing  $\rightarrow$  to highlight the second expression from the negative sign through the 2, that is, almost at the end but not including the last brace.

- Copy and paste this expression to  $y2$  in the same way you did with  $y1$ .

- To set the standard Window variable values, press  $\boxed{F2}$  **Zoom** and select **6:ZoomStd**. The graph of the implicit relation is displayed.

- To draw a tangent line and display the equation, press  $\boxed{F5}$  **Math** and select **A:Tangent**. Enter **4** and press  $\boxed{ENTER}$ .

- To draw the other tangent line and display the equation, repeat the tangent command and press  $\downarrow$  to move to the graph  $y2$ . Enter **4** and press  $\boxed{ENTER}$ .



**Example 4: Symbolic differentiation**

It is important to remember that you have a calculator with a symbolic algebra system. Derivatives can be computed that help investigate rules of differentiation, such as the product rule and the chain rule.

Compute the derivatives for  $\cos(u(x))$ , and  $(u(x))^n$ , and  $u(x) \cdot v(x)$ .

**Solution**

Use the differentiate command with  $u(x)$  and  $v(x)$  as undefined functions.

1. From the Home screen, press  $\boxed{2\text{nd}} \boxed{F6}$  **Clean Up** and select **1:clear a-z** to clear the variable name.

2. Differentiate  $\cos(u(x))$  with respect to  $x$ .

$\boxed{2\text{nd}} \boxed{d} \boxed{2\text{nd}} \boxed{\text{COS}} \boxed{U} \boxed{(} \boxed{X} \boxed{)} \boxed{)} \boxed{,} \boxed{X} \boxed{)} \boxed{\text{ENTER}}$

$$\frac{d}{dx}(\cos(u(x)))$$

$$-\sin(u(x)) \cdot \frac{d}{dx}(u(x))$$

$$\frac{d(\cos(u(x)), x)}$$

3. Differentiate  $(u(x))^n$ .

$\boxed{2\text{nd}} \boxed{d} \boxed{(} \boxed{U} \boxed{(} \boxed{X} \boxed{)} \boxed{)} \boxed{\wedge} \boxed{N} \boxed{,} \boxed{X} \boxed{)} \boxed{\text{ENTER}}$

$$\frac{d}{dx}((u(x))^n)$$

$$n \cdot (u(x))^{n-1} \cdot \frac{d}{dx}(u(x))$$

$$\frac{d((u(x))^n, x)}$$

4. Differentiate  $u(x) \cdot v(x)$  in the same way.

$\boxed{2\text{nd}} \boxed{d} \boxed{U} \boxed{(} \boxed{X} \boxed{)} \boxed{\cdot} \boxed{V} \boxed{(} \boxed{X} \boxed{)} \boxed{,} \boxed{X} \boxed{)} \boxed{\text{ENTER}}$

$$\frac{d}{dx}(u(x) \cdot v(x))$$

$$\frac{d}{dx}(u(x)) \cdot v(x) + \frac{d}{dx}(v(x)) \cdot u(x)$$

$$\frac{d(u(x) \cdot v(x), x)}$$

**Exercises**

For exercises 1 to 8, use  $f(x) = \sqrt{x^2 + 4x}$ .

1. Compute the average rate of change from 2 to 2.001 and from 2 to 2.0001.
2. Compute the numerical derivative at  $x=2$ , using  $h=.001$  and  $.0001$ .
3. Compute the limit of the average rate of change from 2 to  $2+h$  as  $h$  approaches zero.
4. Compute the limit of the average rate of change from  $x$  to  $x+h$  as  $h$  approaches zero.
5. Compute  $f'(x)$ .
6. Compute  $f'(2)$ .
7. Compute the equation of the tangent line at  $x=2$ .
8. Compute  $f''(x)$ .

For exercises 9 to 12, use  $3x^2 - xy + y^2 - 4x + 5y = 10$ .

9. Compute  $\frac{dy}{dx}$ .
10. Compute  $y$  when  $x = -2$ .
11. Compute the slope of the curve at all points where  $x = -2$ .
12. Compute the equations for the tangent lines when  $x = -2$ .
13. Compute the derivative for  $f(x) = \frac{u(x)}{v(x)}$ .