What's My Locus? – ID: 8255

By Lewis Lum

Time required
45 minutes

Activity Overview

In this activity, students will explore the focus/directrix and reflection properties of parabolas. They are led to conjecture each property and then prove them analytically.

Concepts

- *Parabolas as a locus of points*
- *The distance formula in the coordinate plane*
- *Elementary algebra and plane geometry*
- *The derivative as the slope of a tangent line*

Teacher Preparation

This activity may be used by Geometry, Algebra 2, and Elementary Calculus students. A simple derivative is used only in the proof of the reflection property (Problem 6) and can easily be circumvented by using the TI-Nspire angle measurement tool.

- *Students should be familiar with the following geometry concepts and facts: the shape of the graph of a parabola; the formula for calculating the distance between two points in the plane; vertical angles of two intersecting lines are equal, and the altitude at vertex A of an isosceles triangle bisects the angle at A.*
- *The screenshots on pages 2–4 demonstrate expected student results. Refer to the screenshots on page 5 for a preview of the student .tns file.*
- ***To download the .tns file and student worksheet, go to http://education.ti.com/exchange and enter "8255" in the search box.***

Classroom Management

- *This activity is designed to have students explore **individually or in pairs**. However, an alternate approach would be to use the activity in a whole-class format. By using the computer software and the questions found on the student worksheet, you can lead an interactive class discussion about these parabola properties.*
- *Some of the pages are vertically split: G&G on the left and L&S on the right. Calculations that drive the implementations are hidden in the spreadsheet; the "split" is designed to expose only Column A. Caution students to leave the rest of the spreadsheet alone.*
- *The student worksheet is intended to guide students through the main ideas of the activity. It also serves as a place for students to record their answers. Alternatively, you may wish to have the class record their answers on a separate sheet of paper, or just use the questions posed to engage a class discussion.*

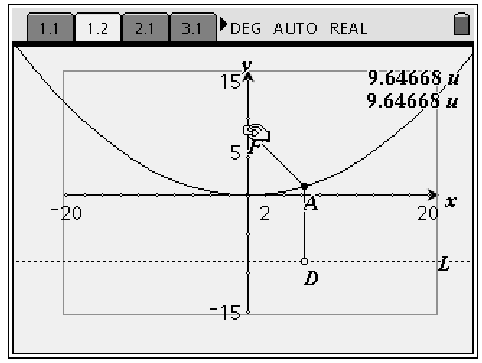
TI-Nspire™ Applications

Graphs & Geometry (G&G), Lists & Spreadsheet (L&S), Notes

The first four problems in this activity lead students through the derivation of the formula for a parabola from its geometric definition. They also aim to test students' understanding of the concepts uncovered by the derivation. Problems 5 and 6 are designed to help students explore the reflection property of parabolas.

Problem 1 – Locus of points equidistant from a fixed point and a fixed line

On page 1.2, students are shown a graph with a point *F*(0, *p*) on the *y*-axis, a point *D* on a line *L* given by the equation $y = -p$, and a point *A* above *D*. They will measure the length of $\overline{FA}$ and $\overline{DA}$, and observe what happens when dragging points *F* (the focus) or *D* (which controls point *A*).



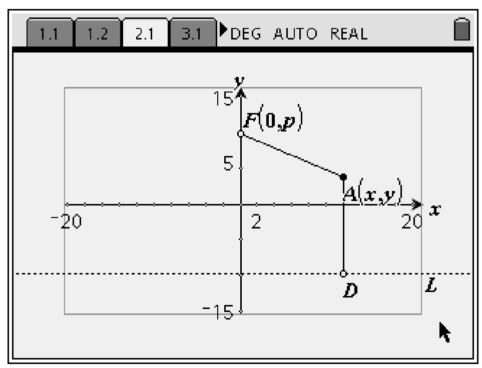
Page 1.2

Solutions

1. The lengths *FA* and *DA* are equal for all points *A*.
2. The locus of points *A* is a parabola with vertex at the origin. If *F* is above the *x*-axis, the parabola opens up; if *F* is below the *x*-axis, it opens down.
3. Answers may vary.

Problem 2 – Derive a formula for the locus of point *A*

The static (points F and D are locked in place) diagram shown in Problem 2 is the same as that in Problem 1, with the points labeled: *F*(0, *p*), *A*(*x*, *y*), and *D*. Students will use this diagram to derive the standard form for the equation of a parabola with focus *F*(0, *p*) and directrix $y = -p$.



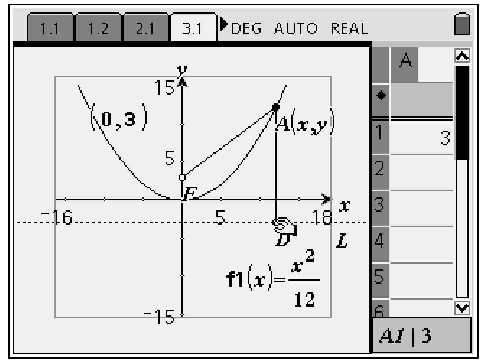
Page 2.1

Solutions

1. $(x, -p)$
2. $FA = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + y^2 - 2py + p^2}$ $\quad DA = \sqrt{(x-x)^2 + (y+p)^2} = \sqrt{y^2 + 2py + p^2}$
3. $$\sqrt{x^2 + y^2 - 2py + p^2} = \sqrt{y^2 + 2py + p^2}$$
$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$
$$x^2 = 4py$$
$$y = \frac{x^2}{4p}$$

Problem 3 – Test your understanding

Before beginning this problem, ensure that students have derived the correct formula from Problem 2; i.e, $y = \frac{x^2}{4p}$. They will use this equation to identify the coordinates of the focus for given parabolas. Page 3.1 enables them to test their answers by entering the value of *p* into cell A1 of the spreadsheet and defining **f1**(*x*) to be the equation of the parabola.



Page 3.1

Solutions

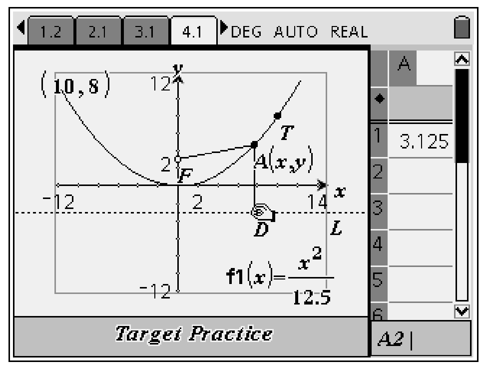
1. *F*(0, 3)

2. *F*(0, –9)

Problem 4 – Target practice

Here, students again use the formula to explore parabolas—this time identifying the equation of a parabola with vertex at the origin that passes through a certain point, *T* (restricted to integer lattice points). Again, they can test their equations using the graph and spreadsheet on page 4.1.

(**Note:** Fractions are sometimes not recognized as numeric input into the spreadsheet. If a student obtains a value of *p* that is a fraction, have them enter it into the spreadsheet in its decimal form. If the fraction's decimal form is a repeating decimal such as 2/3, entering it as "2.0/3" is acceptable.)



Page 4.1

The exercises ask students to generalize to a parabola with the point (*h*, *k*) its vertex (***not*** the origin). The exercises should be solved using paper and pencil, with all work shown.

Solutions

1a. $(h, k+p)$

b. $y = k - p$

c. $y = \frac{(x-h)^2}{4p} + k$

2. Rewrite the equation in vertex form.

$$16y = x^2 - 8x + 96 \qquad \textit{clear the fractions}$$

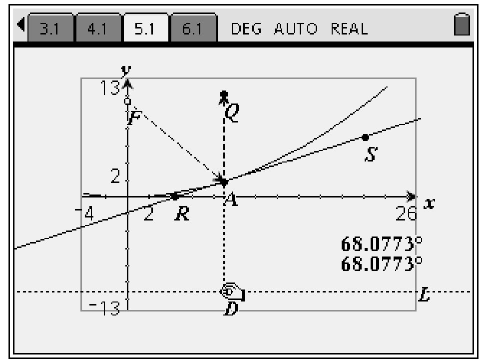
$$16y - 96 \boxed{+16} = x^2 - 8x \boxed{+16} \qquad \textit{complete the square}$$

$$16(y-5) = (x-4)^2$$

$$y = \frac{(x-4)^2}{16} + 5$$

vertex: (4, 5) focus: (4, 9) directrix: $y = 1$

Problem 5 – The reflection property



Page 5.1

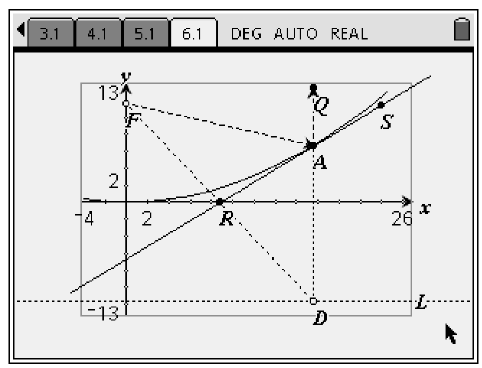
On page 5.1, students are shown a graph of a parabola with focus *F* and tangent line to the parabola at point *A*. They are asked to measure ∠*FAR* and ∠*SAQ*, grab and drag points *F* or *D*, and observe how the angles of incidence and reflection are related.

Note: If point *D* is dragged into the second quadrant, the angle measurement tool calculates the measure of the "wrong angle." Thus, negative *x*-axis has been reduced in length.

Solutions

1. $m\angle FAR = m\angle SAQ$ for all points *A*
2. All rays emanating from the focus will exit the surface of the parabola on a path parallel to the axis of symmetry.
3. Answers will vary.

Problem 6 – A proof of the reflection property[1]



Page 6.1

Page 6.1 displays the same diagram as 5.1, with an additional segment shown. The exercises walk students through a proof of the reflection property (in terms of the diagram, the proof that $m\angle FAR = m\angle SAQ$).

Solutions

1. $\triangle FAD$ is isosceles
2. <u>*For non-calculus students:*</u> Since $m\angle DRA = 90°$, $\overline{FD}$ and the tangent line are perpendicular.

   <u>*For calculus students:*</u> Slope of $\overline{FD} = -\frac{2p}{a}$; Slope of tangent line at point $A = f'(a) = \frac{a}{2p}$.

   The slopes are negative reciprocals, so $\overline{FD}$ and the tangent line are perpendicular.
3. $\overline{AR}$ is an altitude of $\triangle FAD$ at vertex *A*.
4. The altitude of an isosceles triangle bisects the vertex angle, so $m\angle FAR = m\angle DAR$.
5. Vertical angles are congruent, so $m\angle DAR = m\angle SAQ$.
6. By the transitive property of angle congruence, $m\angle FAR = m\angle SAQ$.

[1] Williams, Robert C., *A Proof of the Reflective Property of the Parabola*, American Mathematical Monthly, (1987) 667-668.

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(Student)TI-Nspire File: *PreCalcAct2_WhatsMyLocus_EN.tns*