Hot Coffee

by John F. Mahoney
Banneker Academic High School, Washington, DC
mahoneyj@sidwell.edu

Abstract: This activity is an application of differential equations and slope fields. Students first go through the traditional paper and pencil method of solving a differential equation. They then use the symbolic capacity of their calculator and calculus to solve the equation more easily. At the end they use the calculator's differential equation solver and slope field procedure.

NCTM Principles and Standards:
Algebra standards

a) Understand patterns, relations, and functions
b) analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
c) use symbolic algebra to represent and explain mathematical relationships;
d) judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
e) draw reasonable conclusions about a situation being modeled.

Problem Solving Standard build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof Standard

a) recognize reasoning and proof as fundamental aspects of mathematics;
b) make and investigate mathematical conjectures;
c) develop and evaluate mathematical arguments and proofs;
d) select and use various types of reasoning and methods of proof.

Representation Standard : use representations to model and interpret physical, social, and phenomena.

Key topic: Differential equations and slope fields

Degree of Difficulty: moderate to advanced
Needed Materials: TI-89 calculator

Situation: Starbrick's coffee is very hot. At the coffee shop, its temperature is 200 degrees when first served and its temperature is dropping at the rate of 5 degrees per minute. The temperature of the air in the coffee shop is 65 degrees. According to Newton's law of cooling, the coffee will cool at a rate proportional to the difference between its temperature and the temperature of the air in the shop. The coffee is safe to drink when its temperature is 175 degrees. When will this particular cup of coffee be safe to drink?

This is an example of a differential equation. If y is the temperature of the coffee then we have: $\frac{dy}{dt} = k(y-65)$.

We can solve this differential equation in a variety of ways.

Separation of variables: Some equations, like this one, lend themselves to this technique because all of the parts of the problem which have y's in them can be put on one side of the equation – and the parts with t's can be put on the other side: $\frac{dy}{(y-65)} = kdt$

Let's first consider doing this problem by hand and then we'll take a look at how the TI-89 calculator can help:

We can integrate the left side with respect to y and the right side with respect to t: $\ln(y-65) = kt$.

But we must remember to add a constant of integration: $\ln(y-65) = kt + C$

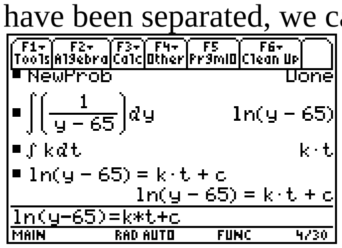
Since we know that when t = 0 (when the coffee is first served) its temperature is 200 degrees, then we have $\ln(200-65) = k*0 + C$ or ln 135 = C

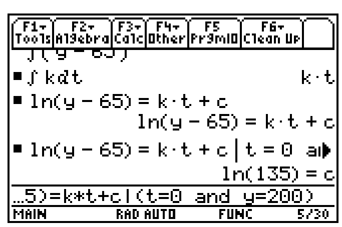
We also know that $\frac{dy}{dt} = -5$ initially so we can substitute into the original equation:

$\frac{dy}{dt} = k(y-65)$ and get -5 = k(200 – 65) or k= -1/27 so our equation is

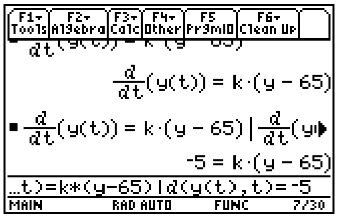
$\ln(y-65) = \frac{-t}{27} + \ln 135$

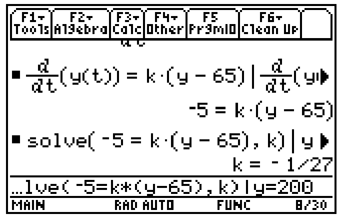
To solve for y, first exponentiate both sides (base e) to get $y - 65 = e^{-t/27 + \ln 135}$ which simplifies to $y = 135e^{-t/27} + 65$. If we solve this equation for t when y = 175, our solution is t= 5.53 min.

Now, turn on your TI-89 and follow these next steps: If we start after the variables have been separated, we can integrate each side and form the resulting equation: 

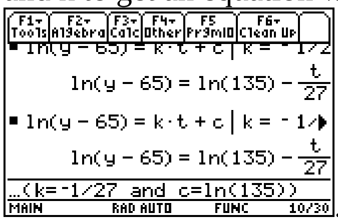
We can use the initial conditions to solve for c: 

and we can use the original differential equation and the initial conditions to solve for k.

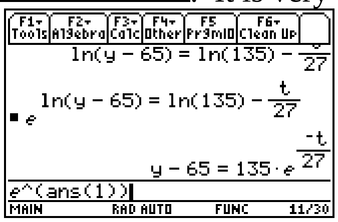


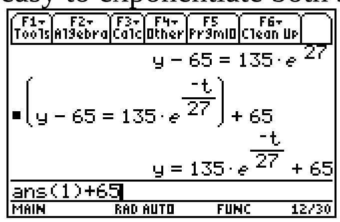


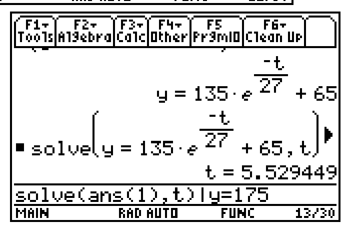
Now we can substitute for both c and k to get an equation with just y's and t's in it:



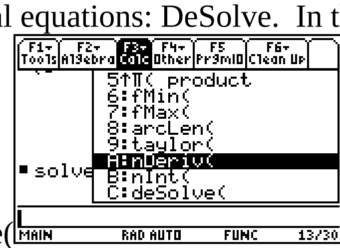
. It is very easy to exponentiate both sides (base e) on the calculator:



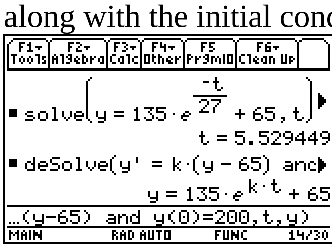


Finally we can solve for t when y = 175: 

That certainly wasn't too hard, but the TI-89 has an even much easier way to solve differential equations: DeSolve. In the calculus menu, scroll down to choice C:deSolve(



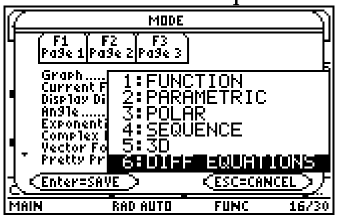
Here we can enter our original differential equation, along with the initial conditions, without separating variables:



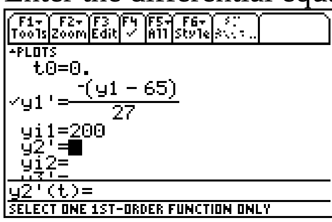
We can now solve for the value of k as we did above.

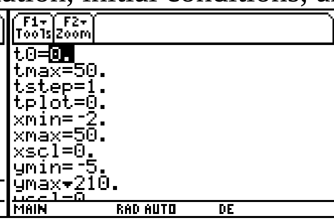
The graphs of differential equations can be represented as slope fields. Let's start with our particular equation: $\frac{dy}{dt} = \frac{-1}{27}(y-65)$

Choose 6: Diff Equations from the Graph part of the Mode menu:



Enter the differential equation, initial conditions, and window parameters as shown:





We can produce the slope field and see the graph of our particular solution. Not surprisingly, we can see that the curve will be asymptotic to the line y = 65 representing the temperature of the coffee house. We can also examine the value of the curve when t = 5.53 minutes to verify that the temperature of the coffee is 175°.