Chapter 7

Probability

In this chapter

*This chapter summarizes NCTM's **Principles and Standards for School Mathematics** for probability and applies those principles in some activity overviews. You will learn how to:*

- *Explore the probability concepts of **most likely** and **least likely**.*
- *Use the calculator to help students develop concepts of probability.*

Overview of probability

The NCTM Principles and Standards for School Mathematics state that there should be an emphasis on the way that probability and statistics are related. Students should engage in experiments to generate data from which ideas about probability will evolve.

Goals for students

In the elementary grades, students' work with ideas of probability should be informal; however by grades 3-5,

> students can consider ideas of chance through experiments---using coins, dice, or spinners---with known theoretical outcomes or through designating familiar events as impossible, unlikely, likely, or certain. .... Through the grades, students should be able to move from situations for which the probability of an event can readily be determined to situations in which sampling and simulations help them quantify the likelihood of an uncertain outcome (NCTM, 51).
>
>

As students examine predictable outcomes such as flipping coins or rolling dice, they see that these events repeated many times form a pattern of distribution.

The ideas about probability in prekindergarten through grade 2 should be informal and based on the children's experiences with rolling dice or spinning spinners. The probability activities for these grades also further develop the idea of numbers. Questions about *most likely* or *least likely* lead to the notions of impossible events and more or less likely events.

At grades 3 – 5 the NCTM Principles and Standards for School Mathematics describes students' abilities to understand and apply the basic concepts of probability as the following:

- describe events as likely or unlikely and discuss the degree of likelihood using such words as *certain*, *equally likely*, and *impossible*;
- predict the probability of outcomes of simple experiments and test the predictions;
- understand that the measure of the likelihood of an event can be represented by a number from 0 to 1.

Students at grades 3 –5 should conduct experiments and collect data from which they can quantify the likelihood of an event. From these experiments, students see how they can predict the frequency of various outcomes even though they cannot determine an individual outcome. The use of **zero** to represent the probability of an impossible event and **one** to represent a certain event, along with the use of fractions to represent other events, should be in the curriculum for students at this level.

Goals for teachers

In order for teachers to provide the curriculum that the NCTM Principles and Standards recommends, the CBMS states that teachers must conduct their own experiments to help them recognize and understand that given equally likely outcomes, the probability of a particular event is equal to the ratio of the number of outcomes defined by the event to the number of total possible outcomes. The document goes on to note that teachers must think through what randomness means and note the difference between predicting an individual event and predicting a pattern of events.

Specifically, the CBMS states that teachers need experience in "probability: making judgments under conditions of uncertainty, measuring likelihood, becoming familiar with the idea of randomness" (CBMS, 23).

Sample activities

It is important to use concrete materials to help students make connections to the concepts of probability, which can be too abstract when taught without these materials. Children can use manipulatives, calculators, and paper and pencil all at the same time to make some of these connections more apparent. Activities help students develop concepts of probability, and concepts from other areas in mathematics.

Activity 1: Multiply or Add to Get a Larger Number?

If you roll three dice and list the rolls as **a**, **b**, and **c**, which of the following is more likely? Make a prediction.

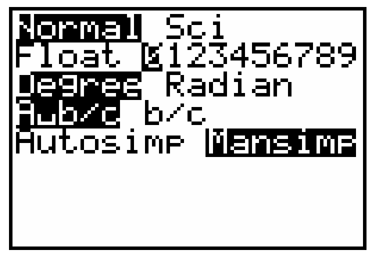
$$a \times b > b + c$$

$$a \times b < b + c$$

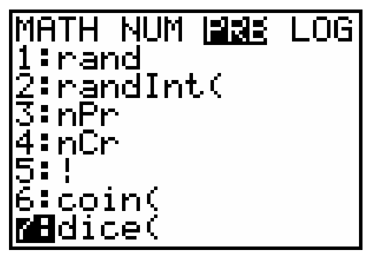
$$a \times b = b + c$$

To explore this problem, you can make use of the TI-73 dice-rolling feature. Rolling dice is performed as follows:

1. Press [MODE] and highlight **0** to set the decimal places to zero. Press [ENTER].



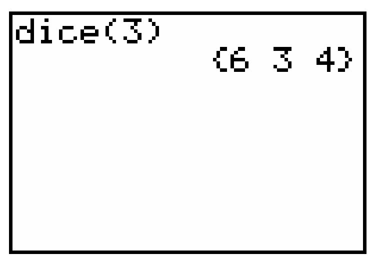
2. Press [MATH]. Press [▶] [▶] to select **PRB.**
3. Press [▼] a few times to select **7:dice(**, and then press [ENTER].



4. The Home screen displays **dice(**. Since you want to have three dice to explore this question, press **3** [)] and then press [ENTER].

   The Home screen displays a random ordered triple from the integers 1 – 6. As you continue to press [ENTER], more triples appear.

   Note*: Your screen may not appear exactly like the one at the right. Your TI-73 may produce other random triples.*



Use the triples on the TI-73 and a table like the one below to record your investigation and to compare ***a*** x ***b*** and ***b*** + ***c***. Roll the dice at least 20 times before answering the questions that follow.

Roll number	**Roll of dice (*a*, *b*, *c*)**	***a* x *b***	***b* + *c***
1	6, 2, 4	12	6
2	3, 6, 3	18	9
3	4, 2, 5	8	7
4			
(and so on)			

Answer questions 1 - 8 as a *learner* of mathematics.

1. **a.** Number of instances a * b > b + c _________

b. Number of instances a * b < b + c _________

c. Number of instances a * b = b + c _________

2. **a.** From the data you collected, which did you determine is most likely?

(a * b > b + c) or (a * b < b + c) or (a * b = b + c)? Why?

b. From the data you collected, which did you determine is least likely?

(a * b > b + c) or (a * b < b + c) or (a * b = b + c)? Why?

3. **a.** Percent of instances a * b > b + c _________

b. Percent of instances a * b < b + c _________

c. Percent of instances a * b = b + c _________

4. **a.** Probability of a * b > b + c _______________

b. Probability of a * b < b + c _______________

c. Probability of a * b = b + c _______________

5. Compile class data to see if the same conclusions are reached as found with individual data sets.

6. Using the TI-73, make circle graphs of your individual data and then of the class data. How do they compare?

7. Using the same triples, would the results be similar if b * c and a + c were compared? Explain. How about a * c and b + c?

8. Would the results be the same if we rolled only two dice and compared the product of the two numbers to the sum of the two numbers? Why or why not?

Answer questions 9 - 13 as a *teacher* of mathematics.

9. Did students make appropriate predictions? Were any students surprised about their results? If so, what surprised them? What did they expect?

10. Were students able to see the relationship between probabilities and the percentages? Did they use the features of the TI-73 to work with fractions, decimals, and percents?

11. How was the creation of circle graphs with the calculator helpful for comparing individual and class data?

12. How did the students respond to using the dice feature on the TI-73 for this activity?

13. Did students work with the data comfortably or did they experience frustration? Were you able to identify what may have caused any student frustration?

Answers and comments

This activity explores the probability concepts of *most likely* and *least likely*, while giving students experience with generating and organizing data. In addition, students develop more number sense as they compare multiplication and addition of whole numbers and rational numbers. The main goal of this activity should be to have students collect experimental data from which to draw conclusions.

Questions **1 – 5**

Probabilities and percents will vary, but students will likely agree with the following results:

$a * b > b + c$ (Most likely) $a * b < b + c$ $a * b = b + c$ (Least likely)

Students ready to explore this problem looking at the theoretical probability should be encouraged to draw a model from which they can determine the actual probability. There are six choices for each die, therefore there are 6 x 6 x 6 = 216 entries to examine. The computed probabilities are given below.

$$P(a * b > b + c) = \frac{140}{216} \approx 65\% \qquad P(a * b < b + c) = \frac{63}{216} \approx 29\% \qquad P(a * b = b + c) = \frac{13}{216} \approx 6\%$$

Following is a sample model showing dice rolls beginning with **1** or **2**.

a*b > a+b is represented by >

a*b < a+b is represented by <

a*b = a+b is represented by =

Dice Roll	Outcome	Dice Roll	Outcome		Dice Roll	Outcome
1, 1, 1	<	1, 3, 1	<		1, 5, 1	<
1, 1, 2	<	1, 3, 2	<		1, 5, 2	<
1, 1, 3	<	1, 3, 3	<		1, 5, 3	<
1, 1, 4	<	1, 3, 4	<		1, 5, 4	<
1, 1, 5	<	1, 3, 5	<		1, 5, 5	<
1, 1, 6	<	1, 3, 6	<		1, 5, 6	<
1, 2, 1	<	1, 4, 1	<		1, 6, 1	<
1, 2, 2	<	1, 4, 2	<		1, 6, 2	<
1, 2, 3	<	1, 4, 3	<		1 , 6, 3	<
1, 2, 4	<	1, 4, 4	<		1, 6, 4	<
1, 2, 5	<	1, 4, 5	<		1, 6, 5	<
1, 2, 6	<	1, 4, 6	<		1, 6, 6	<

Dice Roll	Outcome	Dice Roll	Outcome		Dice Roll	Outcome
2, 1, 1	=	2, 3, 1	>		2, 5, 1	>
2, 1, 2	<	2, 3, 2	>		2, 5, 2	>
2, 1, 3	<	2, 3, 3	=		2, 5, 3	>
2, 1, 4	<	2, 3, 4	<		2, 5, 4	>
2, 1, 5	<	2, 3, 5	<		2, 5, 5	=
2, 1, 6	<	2, 3, 6	<		2, 5, 6	<
2, 2, 1	>	2, 4, 1	>		2, 6, 1	>
2, 2, 2	=	2, 4, 2	>		2, 6, 2	>
2, 2, 3	<	2, 4, 3	>		2 , 6, 3	>
2, 2, 4	<	2, 4, 4	=		2, 6, 4	>
2, 2, 5	<	2, 4, 5	<		2, 6, 5	>
2, 2, 6	<	2, 4, 6	<		2, 6, 6	=

Question **6**

To create a circle graph, first enter data into lists. It will be necessary to have a category list along with a list indicating the numerical value of each of the categories. Pressing [2nd] [PLOT] allows you to select **Plot1** and turn it on. From there you can make the appropriate selections and then press [GRAPH]. Comparisons will vary, but the graphs should look similar.

Question **7**

The probabilities would be the same because the comparisons are essentially the same.

Question **8**

The results would be different because there would be only 36 possibilities. The product, *a* x *b*, and the sum, *a* + *b*, would be equal in only one instance, 2 + 2 and 2 x 2. Hence, P(a + b = a * b) = $\frac{1}{36}$.

The sum would be larger than the product whenever a 1 is added to a number compared to multiplying 1 times a number. That is, whenever at least one die has a 1 showing, the sum will be larger than the product. This happens 11 times, so P(a + b > a * b) = $\frac{11}{36}$.

In all other cases, the product will be larger than the sum, so P(a * b > a + b) = $\frac{24}{36}$.

a*b >a+b is represented by >

a*b < a+b is represented by <

a*b = a+b is represented by =

Dice Roll	**Out-come**	**Dice Roll**	**Out-come**	**Dice Roll**	**Out-come**	**Dice Roll**	**Out-come**	**Dice Roll**	**Out-come**	**Dice Roll**	**Out-come**
1, 1	<	1 ,2	<	1, 3	<	1, 4	<	1, 5	<	1, 6	<
2, 1	<	2, 2	=	2, 3	>	2, 4	>	2, 5	>	2, 6	>
3, 1	<	3, 2	>	3, 3	>	3, 4	>	3, 5	>	3, 6	>
4, 1	<	4, 2	>	4, 3	>	4, 4	>	4, 5	>	4, 6	>
5, 1	<	5,2	>	5, 3	>	5, 4	>	5, 5	>	5, 6	>
6, 1	<	6, 2	>	6, 3	>	6, 4	>	6, 5	>	6, 6	>

Question **9**

Students should be encouraged to make reflective predictions prior to rolling the dice and recording data. Students tend to regard predictions as guesses and do not examine questions using their prior knowledge to make a prediction. Strategies that can help develop reflective learners should be incorporated whenever possible as students engage in activities with or without the use of calculators.

Question **10**

Exploring probability is an excellent opportunity to engage students in work with fractions, decimals, and percents. These rational number features on the TI-73 along with the probability features make this an excellent tool for developing mathematical concepts. Some students may need additional guidance in working with these features on the TI-73.

Question **11**

Since probabilities are values between and including 0 and 1, a circle graph showing part of a whole is an excellent representation to use to display this data, and it is quite easily constructed on the TI-73. Be sure to have students use both the number and the percent values on the circle graph.

Question **12**

Students will likely be intrigued with the dice feature and should be given some time to explore various number of inputs for dice rolls, especially if this is their first experience with this feature on the TI-73.

Question **13**

This activity allows students to make a prediction and begin collecting data to compare their experimental results with their predictions. It also provides sufficient challenges for students to continue the activity and work with theoretical probabilities. Individual students may reach frustration levels at different points, however, creating groups of students at different levels of understanding should benefit all students.

Activity 2: Flip Five

If you flipped five coins:

What is the probability of getting exactly 3 heads?

What is the probability of getting at least 3 heads?

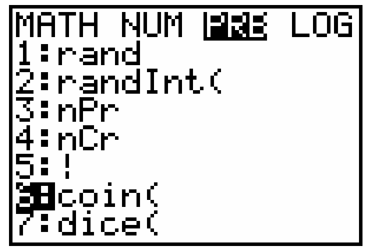
What is the probability of getting at most (no more than) 3 heads?

Make your predictions for each question before flipping the coins.

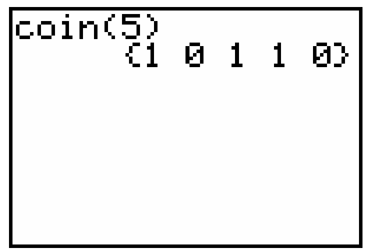
To explore this problem, you can use the **coin(** feature on the TI-73.

1. Press [MATH]. Press [▶] [▶] to select **PRB.**

2. Press [▼] a few times to select **6:coin(**, and then press [ENTER].



3. The Home screen displays **coin(**. Press **5** [)] to simulate flipping five coins, and then press [ENTER].



The Home screen displays a sequence of five numbers consisting of either a **1** or a **0**. You can choose to use the **1** to represent heads and the **0** to represent tails. Therefore, the display at the right would represent three heads and two tails.

Use a table similar to the one below to record the coin flips for five coins. Record at least 20 flips.

Coin Toss	**Number of Heads**
1	
2	
3	
(and so on)	

Answer questions 1 - 7 as a *learner* of mathematics.

1. From the data you recorded in the table, what is the experimental probability of getting exactly 3 heads? At least 3 heads? At most (no more than) 3 heads?

2. Repeat the experiment. What are your results this time?

3. How close were your predictions to your experimental results?

4. **a.** Combine the class results and determine the experimental probability for getting exactly 3 heads, at least 3 heads, or no more than 3 heads.

b. Express the experimental probabilities as decimals and percents as well as fractions.

5. When tossing three coins, what are all the possible outcomes? What is the probability of each of the outcomes? Generate some data to see how your experimental probabilities compare with the theoretical probabilities.

6. List all the possible outcomes for tossing four coins. What is the probability of each of the outcomes?

7. **a.** What are all the possible outcomes for tossing five coins? List the outcomes and the theoretical probabilities for each.

b. How do the class results compare with the theoretical probabilities for flipping five coins? How might your experimental probability come closer to the theoretical probability?

Answer questions 8 - 11 as a *teacher* of mathematics.

8. How accurate were the students' predictions?

9. Did students find the coin feature useful in conducting this probability experiment? In what ways did the TI-73 enhance this activity?

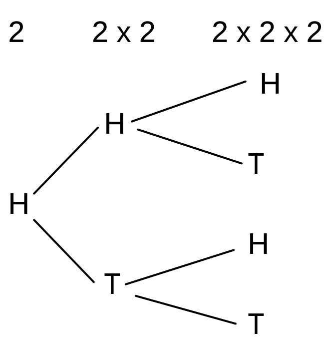
10. What strategies did students use to identify the possible outcomes for flipping three, four, and five coins?

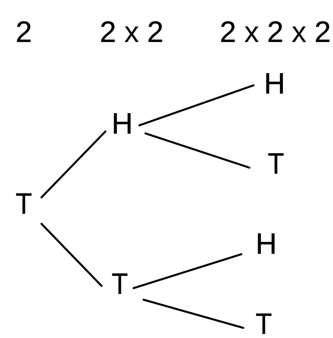
11. Do you think students feel more confident working with fractions, decimals, and percents when expressing probabilities and understanding their values?

12. How was this activity helpful to students in developing their understanding of theoretical and experimental probabilities?

Answers and comments

Prior to engaging students in this activity of flipping five coins, there should be some preliminary discussion about the probability of getting a heads or tails when one coin is tossed. Have students record results from flipping two coins to determine the sample space (HH, HT, TH, TT). A tree diagram could be used to help students see that as the number of coins that are flipped is increased, the sample space increased by a power of two. (H = Heads, T = Tails)





After students have recorded results from flipping two coins, it would be appropriate to pose the questions related to flipping five coins and have them make predictions. At this point, the focus should be on conducting the investigation of flipping five coins and gathering empirical evidence. This empirical evidence should be examined and discussed before theoretical probabilities are determined.

Questions **1 - 4**

Since the sample space for flipping five coins is 2 x 2 x 2 x 2 x 2 or 32, students' experimental probabilities should be close to the theoretical probability of:

Exactly 3 heads = $\frac{10}{32}$, 0.3125, 31.25%

At least 3 heads = $\frac{16}{32}$, 0.5, 50%

At most 3 heads = $\frac{26}{32}$, 0.8125, 81.25%

Question **5**

(P = Probability, H = Heads, T = Tails)

P(HHH) = $\frac{1}{8}$, P(HHT) = $\frac{3}{8}$, P(TTH) = $\frac{3}{8}$, P(TTT) = $\frac{1}{8}$

Question **6**

P(HHHH) = $\frac{1}{16}$, P(HHHT) = $\frac{4}{16}$, P(HHTT) = $\frac{6}{16}$, P(HTTT) = $\frac{4}{16}$, P(TTTT) = $\frac{1}{16}$

Question **7. a.**

P(HHHHH) = $\frac{1}{32}$, P(HHHHT) = $\frac{5}{32}$, P(HHHTT) = $\frac{10}{32}$, P(HHTTT) = $\frac{10}{32}$,

P(HTTTT) = $\frac{5}{32}$, P(TTTTT) = $\frac{1}{32}$

Question **7. b.**

At this point, students should recognize that the more times an experiment is done, the closer the empirical probability comes to the theoretical probability.

Question **8**

Accuracy of predictions will vary according to students' understanding of the probabilities of flipping coins and their willingness to reflect on the questions prior to conducting the experiment. Observations of students' predictions can provide you with a pre-assessment of students' level of understanding.

Question **9**

Introductory investigations with flipping coins should be done with actual coins with a smaller number of coins and flips. However, for gathering large amounts of empirical data, using the coin feature on the TI-73 can save a significant amount of time.

Question **10**

By using a tree diagram along with a listing of possible outcomes, students should be able to see the pattern of powers of 2 for determining outcomes for flipping multiple coins.

Question **11**

Observe students' use of fractions, decimals, and percents to determine their confidence and comfort with each representation. Encourage students' use of the fraction features of the TI-73 or TI-15 to develop more confidence with the different representations of rational numbers in working with probabilities.

Question **12**

Most students are comfortable with flipping two coins and comparing experimental probability with theoretical probability. By challenging students with this activity of flipping five coins, they are given the opportunity to transfer this understanding to another situation.

Using calculators to assess probability

Adapted from: Van Zoest, L. R. & Walker, R. K. (1997). Racing to understand probability. *Mathematics Teaching in the Middle School*, 3(2), pp. 162-170.

Eleven students entered into a race with their scooters. They had the numbers 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 taped on their t-shirts. To simulate the race, two large dice are rolled and the sum is recorded. The student whose number matches the sum of the dice moves forward one space on the track below. The winner is the first scooter to cross the finish line.

Student #	START				Track					FINISH LINE	
2											
3											
4											
5											
6											
7											
8											
9											
10											
11											
12											

1. Which scooter do you predict has the best chance of crossing the finish line first?

2. Conduct a simulation using the TI-73 to see if your prediction is correct. Keep a record to justify your answer. How can you use the number combinations to determine which scooter will most likely cross the finish line first?

Answers and comments

Probability lends itself well to performance assessment items that use calculators. In this way, the concepts of probability are being assessed rather than the calculations necessary to interpret what is occurring with empirical and theoretical probability. This assessment asks students to engage in problem solving about probability questions in a context of a race with which children can easily identify.

Question **1**

Asking students to predict which scooter has the best chance of crossing the line first is a good leading question to assess the level of students' understanding.

Question 2

Examining students' work should provide information as to which students have an in-depth understanding of the probabilities associated with rolling two dice. Some students may easily identify those numbers with more combinations for their sum as the most likely, while other students rely on the simulation to identify likely candidates.

Activity overviews for K-6: probability

Analyzing Number Cube Sums (Schielack, Jane F. and Chancellor, Dinah, *Uncovering Mathematics with Manipulatives and Calculators Levels 2 & 3*, Texas Instruments, 1995.)

Students should do the *Number Cube Sums* activity (discussed below) before completing this activity so they will have already explored experimental probabilities. Using number cubes of two different colors, students will be able see the ways of rolling two number cubes to get each possible sum of 2 through 12. In this activity students will extend their understanding of theoretical probability and patterns.

Collect All Ten to Win! (Nast, Melissa, ed. *Discovering Mathematics with the TI-73: Activities for Grades 5 and 6*, Texas Instruments, 1998.)

Students who are collectors of cards or toys will identify with this activity and will likely gain a better understanding of the probability of collecting a whole set. This activity simulates a contest that requires the participants to collect 10 different tokens found in the bottom of specially marked cereal boxes.

Does One = One? (Schielack, Jane F. and Chancellor, Dinah, *Uncovering Mathematics with Manipulatives and Calculators Levels 2 & 3*, Texas Instruments, 1995.)

Students will examine a regular six-sided die and the possible outcomes for rolling it one time. They will be primarily working with theoretical probability as they explore fractions and decimals with a calculator. However, this is a good activity for students who need to develop further understanding of the way we use experimental and theoretical probability.

Heads Up (Browning, Christine and Channell, Dwayne, *Graphing Calculator Activities for Enriching Middle School Mathematics*, Texas Instruments, 1997.)

This activity lets students work with some of the important concepts in probability such as *impossible*, *certain*, *likely*, and *not likely*. Students toss coins and also use the random number generator (or the coin feature) on the calculator to conduct a simulation of coin tossing.

Hey, That's Not Fair! (Or is it?) (Nast, Melissa, ed. *Discovering Mathematics with the TI-73: Activities for Grades 5 and 6*, Texas Instruments, 1998.)

Because students are very concerned about fairness, this is an activity with which they will likely identify. Students will use the calculator to simulate dice rolls to play two different games. From examining their data, they will decide if the games give each player an equally likely chance of winning.

Number Cube Sums (Schielack, Jane F. and Chancellor, Dinah, *Uncovering Mathematics with Manipulatives and Calculators Levels 2 & 3*, Texas Instruments, 1995.)

This is a good activity for engaging students in investigating the possible combinations that can be made with the numbers on two different number cubes. By recording and analyzing their data, students can determine experimental probabilities using fractions and decimals.

Picturing Probabilities of Number Cube Sums (Schielack, Jane F. and Chancellor, Dinah, *Uncovering Mathematics with Manipulatives and Calculators Levels 2 & 3*, Texas Instruments, 1995.)

Students should do the *Number Cube Sums* activity (discussed above) before beginning this activity. Students will use ideas of ratio and proportion to investigate various ways to make a circle graph. The graph will display the probabilities of the different sums that can be generated with two number cubes. By using the graphing capabilities of the TI-73 to produce a circle graph, this activity can be even richer.

Spin Me Along (Schielack, Jane F. and Chancellor, Dinah, *Uncovering Mathematics with Manipulatives and Calculators Levels 2 & 3*, Texas Instruments, 1995.)

Students explore probability and patterns in fractions and decimals by spinning three spinners and recording and analyzing the results. By changing the sizes of the sections on the spinners, students can see how the likelihood of the outcomes changes with the different fractional divisions of the circles.

Tiles in a Bag (Schielack, Jane F. and Chancellor, Dinah, *Uncovering Mathematics with Manipulatives and Calculators Levels 2 & 3*, Texas Instruments, 1995.)

Students draw tiles from a bag containing different colored tiles to guess how many tiles of each color are in the bag. They record their results using both fractions and decimals. This activity lets students explore the concept of sample space.