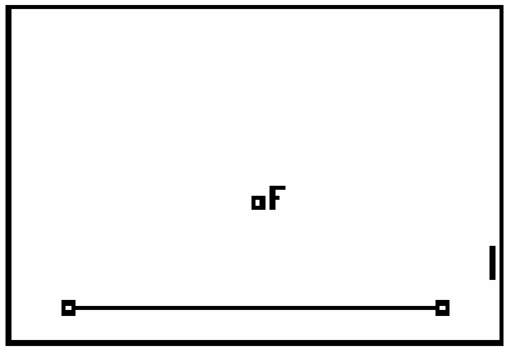
Conics as a Locus of Points
A *Cabri® Jr.* Activity

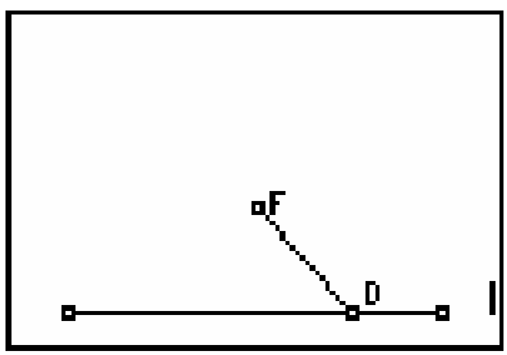
Exploration 1

- Open a new *Cabri® Jr.* file and select **Hide/Show** (**F5**), select **Axes** from the submenu and press **ENTER** to hide the coordinate axes, if necessary.
- Draw a horizontal **Segment** (**F2**) and a **Point** (**F2**) **F** not on the segment.

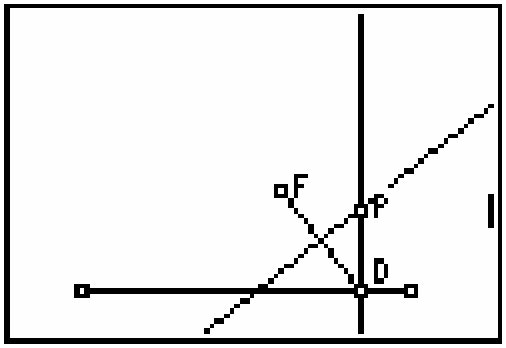


The point is called the *focus* and the line is the *directrix* of a parabola.

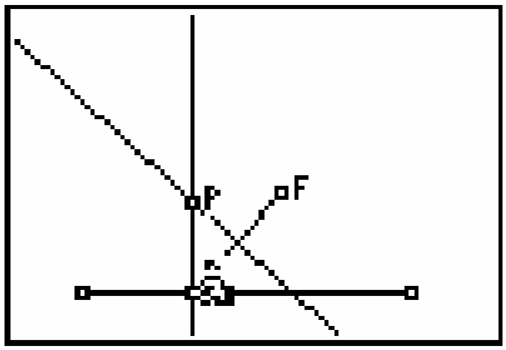
- Construct a **Segment** from the focus **F** to a point **D** on the directrix. (It is not necessary to draw a point on the segment first, simply place the pencil on the segment when drawing the second endpoint.)



- Use the **Perp. Bis.** (**F3**) tool to construct the perpendicular bisector of segment **FD**.
- Through point **D** on the directrix, draw a line perpendicular to the directrix.
- Use the **Point Intersection** (**F2**) tool to construct point **P** at the intersection of the perpendicular line through **D** and the perpendicular bisector of **FD**.

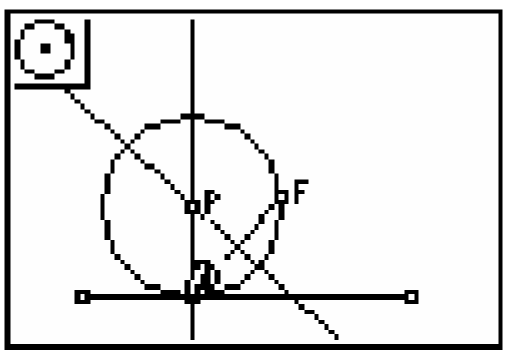


- Drag point **D** along the directrix and observe the path of **P**.



Point **P** is equidistant from the focus **F** and the directrix, which is illustrated in the steps below.

- Draw a **Circle** centered at point **P** passing through point **D** and observe where point **F** is locate

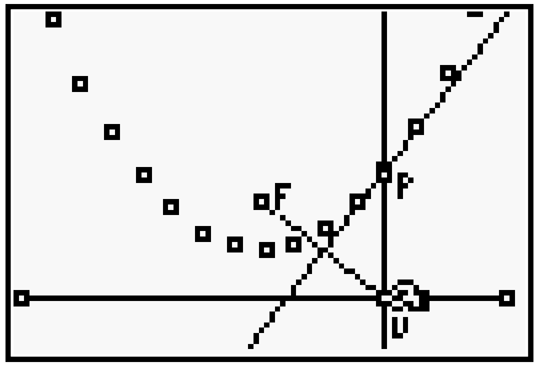


- To show the property that **P** is equidistant to **D** and **F,** complete Δ**PDF** by drawing segment **FP**.

Explain what type of triangle ΔPDF is and why this demonstrates that the points F and D are equidistant to point P.

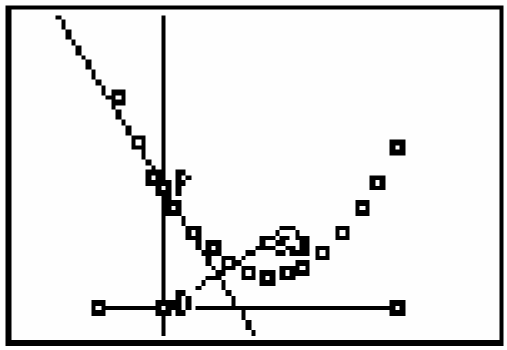
Illustrate the path of point **P** as point **D** moves along the directrix.

- Use the **Hide/Show** tool (**F5**) to hide the circle (and triangle if drawn).
- Execute the **Locus** tool (**F3**) by first selecting the point **P** (trace object) and then point **D** (driving object).
- Drag the point **D** right and left to see that the point **P** moves along the locus points.



Perform sufficient exploration to demonstrate the fact that the point **P** travels long a parabolic path.

- Drag point **F** around the screen.

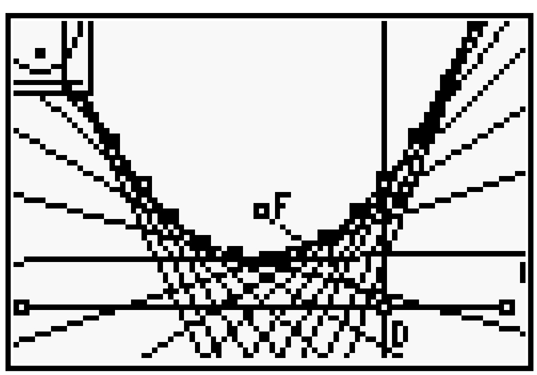


Explain what happens when the focus is moved further from or closer to the directrix.

- Use the **Clear Object** tool (**F5**) to clear the locus of points.

The *envelope* of the perpendicular bisectors by is created in the steps below.

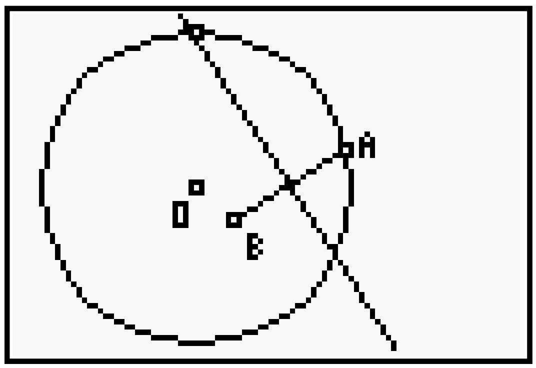
- Engage the **Locus** tool (**F3**)
- Select the perpendicular bisector of segment **FD** as the trace object and the point **D** as the driving object.



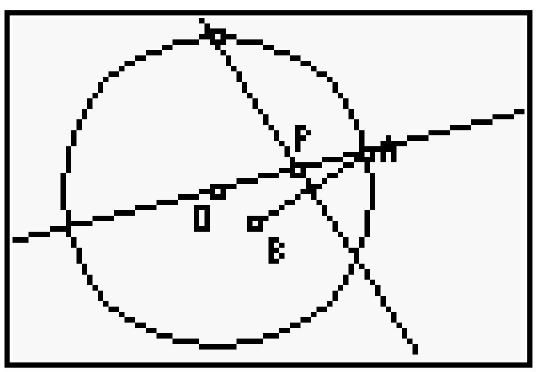
Exploration 2

In the construction of the parabola in Exploration 1, the path of the point **D** along the directrix (a line segment) drove the motion of the locus point. The following illustrates what would happen if the path of the driving point were a circle instead of a segment.

- **Clear:All (F5)** previous objects.
- Construct a **Circle** (**F2**) on the screen and label its center **O**.
- Use **Point:Point On** (**F2**) to construct point **A** on the circle (other than the radius point of the circle).
- From point **A** construct a **Segment** (**F2**) to a point **B** inside the circle.
- Construct the **Perpendicular Bisector** (**Perp. Bis F3**) of the segment **AB**.



- Draw the line **OA**.
- Create the point of intersection of **OA** and the perpendicular bisector, and label it point **P**.

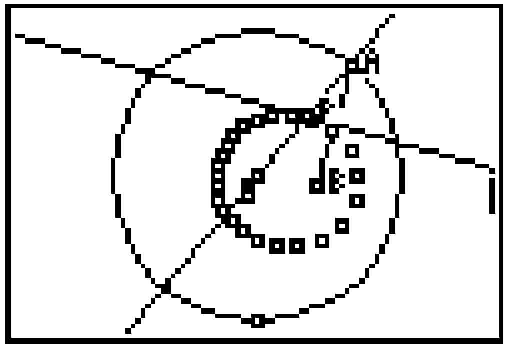


- Use the **Animate** tool (**F1**) to move the point **A** around the circle.

Describe the path of point P.

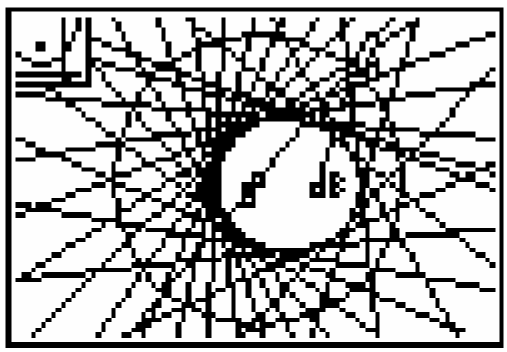
- Press ⌂ **ENTER** to stop the animation.

- Use the **Locus** tool (**F3**) to illustrate the path of **P** by selecting **P** as the trace object and **A** as the driving object.

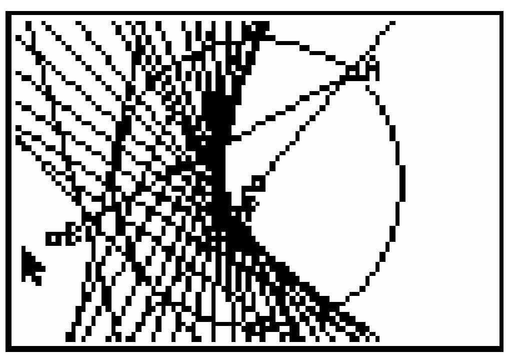


What do the points O and B represent for this curve?

- **Clear** (**F5**) the locus of points.
- Use the **Locus** tool (**F3**) to draw the envelope of the perpendicular bisectors using the perpendicular bisector as the trace object and the point **A** as the driving object.



- Drag the point **B** so that it is outside the circle.

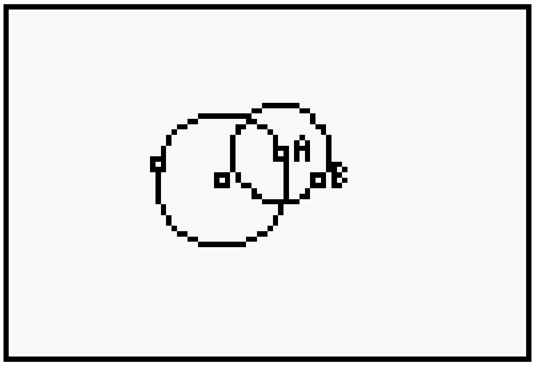


Explain how is the curve changed.

Exploration 3

In Exploration 2 the locus was created when one end of a segment moved around a circular path. What would happen if the segment was replaced by a second circle through a fixed point?

- **Clear All** the objects on the screen.
- Construct a **Circle** (**F2**).
- Create **Point** (**F2**) **B** outside the circle.
- Construct a second **Circle** with the center on the first circle (not the radius point of the first circle) through point **B**.



- **Animate** point **A** on the first circle.

What shape is outlined by the moving circle as the point A travels around the circle?

- Use the **Locus** tool (**F3**) to construct the envelope of the circles to help visualize the shape. The trace object is the circle that contains point **B** and the driving object is point **A**.
- Explore how the locus changes when point **B** is on and inside the circle that contains **A**.

Teaching Notes

1. Students usually encounter the definition of a parabola from a functional viewpoint as the graph of a secondary degree polynomial function, $f(x) = ax^2 + bx + c$. A different way to approach a parabola is through one of its geometric definitions.

   - A parabola is the locus of all points equidistant from a point (the focus) and a line (the directrix).

   The construction uses this alternate definition to "build" the parabola as a locus of points. Measuring the distance from the point of intersection (**P**) to the focus and to the directrix will show that the definition of a parabola is satisfied**.** As point **D** moves along the directrix, **ΔPDF** is always isosceles because of the properties of the perpendicular bisector of its base. This means that **PF = PD**, or that point **P** is equidistant from both a point and a line at the same time.

   An alternate geometric definition of a parabola is demonstrated when a circle is drawn centered at point **P** with radius **PD**. The parabola is defined to be the locus of the center of the circle passing through a fixed point (the focus) and tangent to a fixed line (the directrix). The lines drawn to construct point **P** (the center of the circle) locate the center of the circle by bisecting a chord of the circle and drawing a line perpendicular to a tangent line to the circle at a point of tangency.

2. Students should observe that as the focus **F** moves closer to the directrix, the parabola appears narrower (Figure 1). This is the same graphical effect seen when the value of $\boldsymbol{a}$ in the function $\boldsymbol{y = ax^2}$ is made larger causing a vertical stretch. When the focus is farther away from the directrix, the parabola appears wider, as seen when the value of $\boldsymbol{a}$ in $\boldsymbol{y = ax^2}$ is made smaller causing a vertical compression in the graph (Figure 2). And when the focus is below the directrix, the parabola opens downward as when the value of $\boldsymbol{a}$ in $\boldsymbol{y = ax^2}$ is less than zero (Figure 3).

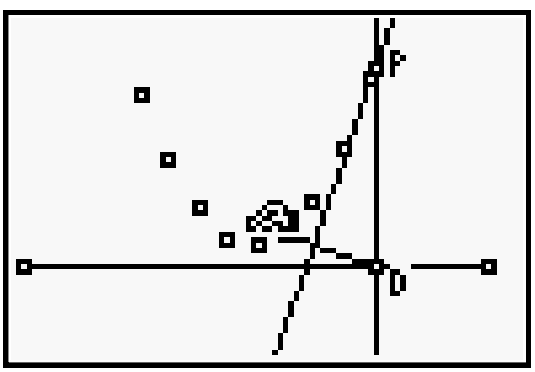


Figure 1

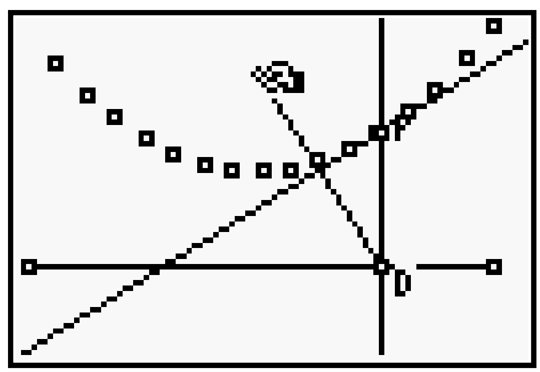


Figure 2

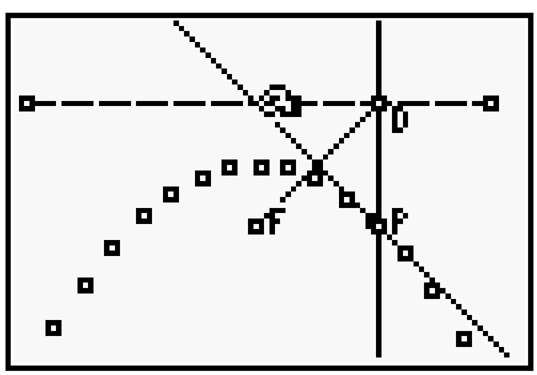


Figure 3

3. If students have studied conic sections, then they will recognize the curves created as loci.

    - An ellipse is the locus of points such that the sum of the distances from the two points (the foci) is a constant.

    - A hyperbola is the locus of points such that the difference of the distances from the two points (the foci) is a constant.

    The locus of point **P** is an ellipse when the point **B** is inside the circle (Figure 4). The points **O** and **B** are the foci. The envelope of the perpendicular bisectors of the ellipse is shown in Figure 5. When the point **B** is outside the circle the locus is a hyperbola (Figure 6).

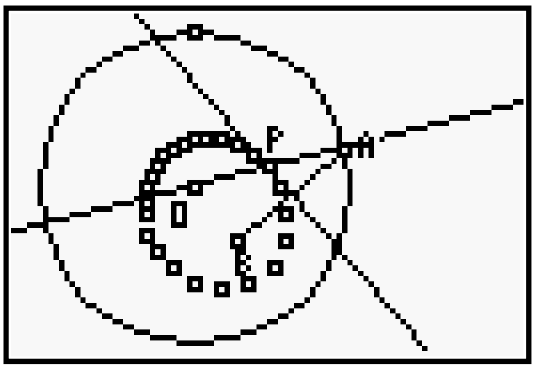


Figure 4

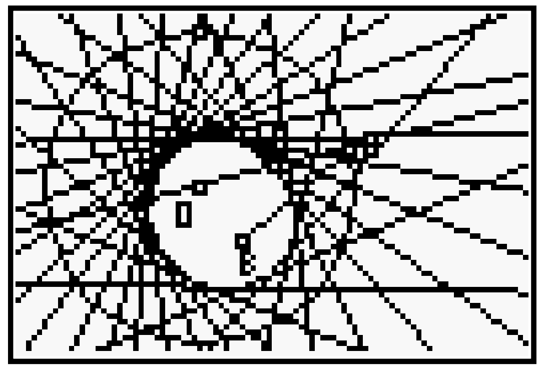


Figure 5

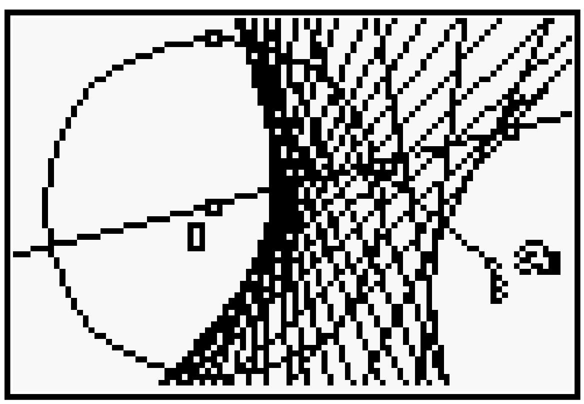


Figure 6

4. The envelope of the circle is a limacon with a loop if the point **B** is outside the circle (Figure 7), a cardioid if point **B** is on the circle (Figure 8), and a limacon with out a loop if point **B** is inside the circle (Figure 9).

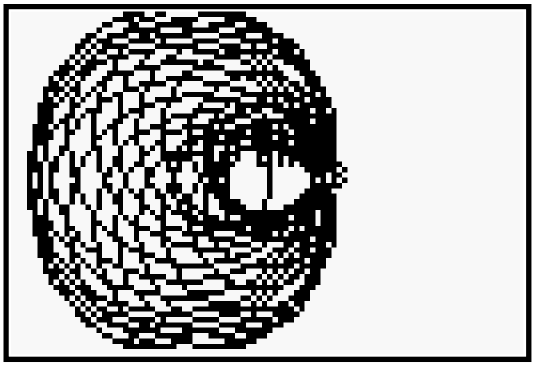


Figure 7

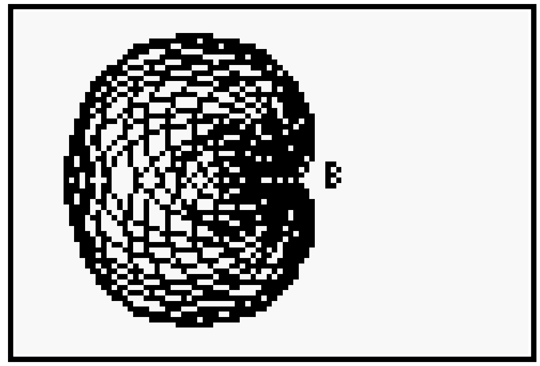


Figure 8

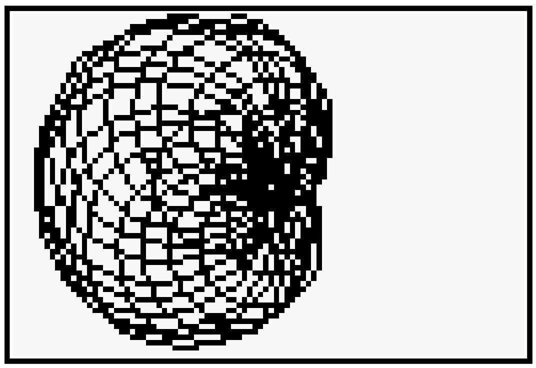


Figure 9