Time Derivatives – ID: 9537

By Marion Glasby

Time required
45 minutes

Activity Overview

In this activity, students will learn how to find velocity and acceleration and identify when the object is at rest, accelerating, and decelerating. In addition, the students will work with related rate problems, exponential growth and decay, and cooling problems.

Concepts

- *Calculating the velocity and acceleration at a given time for a moving object*
- *Identifying the intervals in which a moving object is moving at a constant speed, accelerating, or decelerating*
- *Applying the chain rule to solve problems involving related rates*
- *Using the derivative to solve problems in models of exponential growth and decay*

Teacher Preparation

- *This investigation uses **fMax** to answer a question. Students will have to restrict the domain to get the desired result. Students should be able to graph and take derivatives on their own.*
- *The screenshots on pages 2–5 demonstrate expected student results. Refer to the screenshots on pages 6 and 7 for a preview of the student TI-Nspire document (.tns file).*
- ***To download the student and solution .tns files and student worksheet, go to education.ti.com/exchange and enter "9537" in the quick search box.***

Classroom Management

- *This activity is designed to be **student-centered** with the teacher acting as a facilitator while students work cooperatively. The student worksheet is intended to guide students through the main ideas of the activity and provide a place to record their observations.*
- *Students will need to be able to enter the functions and use the commands on their own.*
- *The ideas contained in the following pages are intended to provide a framework as to how the activity will progress. Suggestions are also provided to help ensure that the activity is completed successfully.*
- *The TI-Nspire solution document CalcAct41_TimeDeriv_Soln_EN.tns shows the expected results of working through the activity.*

TI-Nspire™ Applications

Calculator, Graphs & Geometry, Notes

Introduction

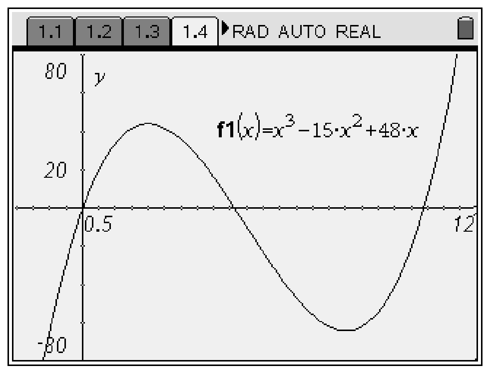
The main concept in this activity, time derivatives, relates to using time, *t*, as the basic underlying variable. For example, let the function be a position function. **Velocity, *v*(*t*),** is the rate of change of distance, so it is the first derivative of the position function. **Acceleration, *a*(*t*),** is the time rate of change of the velocity (Are we getting faster or slower?), so it is the first derivative of velocity, as well as the second derivative of the position function.

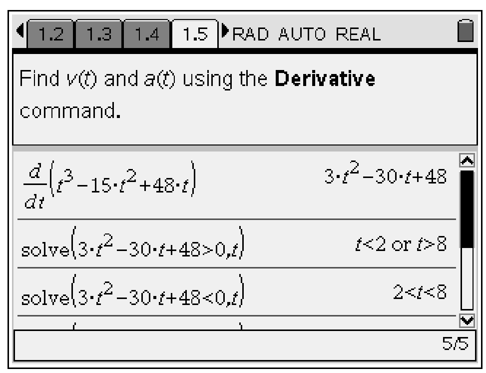
If the function relates rates (such as a growth problem, decay problem, or cooling problem), then the function is a relationship among different variables, but all the variables change with respect to time.

The distance or position function is directional. Hence moving right or up on the coordinate axes is positive and moving down or left is negative. Thus we can have a negative velocity (a ball is dropping or a car is heading west or south) or a positive velocity (a ball is thrown up in the air or a car is heading east or north).

Problem 1 – Velocity and acceleration of position functions

Student will graph the function $s(t)=t^3 - 15t^2 + 48t$. When entering the function, they will need to use *x* instead of *t*. Students can use the **Derivative** and **Solve** commands to help find the answers.





- $v(t) = s'(t) = 3t^2 - 30t + 48$

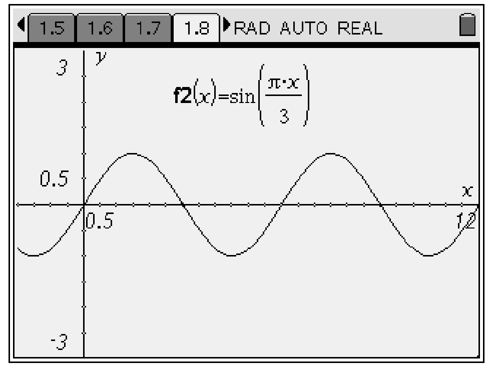
To find when the velocity is negative, positive and at rest, student will need to factor the velocity function. Remind students that $t \geq 0$.

- $v(t) > 0$ when $t < 2$ or $t > 8$
  $v(t) < 0$ when $2 < t < 8$
  $v(t) = 0$ when $t = 2$, $t = 8$

To find when the acceleration is negative, positive, or constant, students will need to find the zeros of the acceleration function.

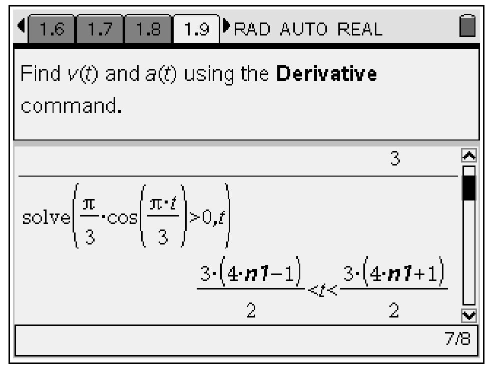
- $a(t) = v'(t) = s''(t) = 6t - 30$
- $a(t) > 0$ when $t > 5$
  $a(t) < 0$ when $t < 5$
  $a(t) = 0$ (constant speed) when $t = 5$

Students are to graph the function $s(t) = \sin\left(\frac{\pi t}{3}\right)$.



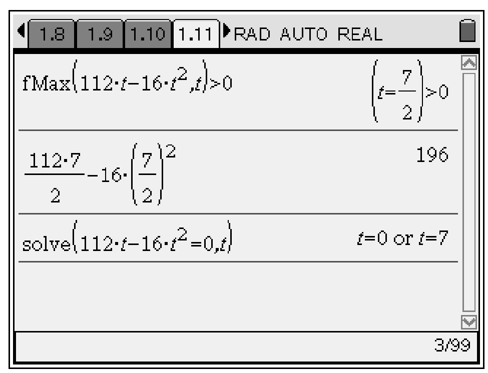
- $v(t) = s'(t) = \frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right)$

When the students use the **Solve** command, a series of terms will appear. Remember: cos(*t*) > 0 in the interval $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$. Adjusting for the period, $\cos\left(\frac{\pi t}{3}\right) > 0$ on (–1.5, 1.5).



- *v(t)* > 0 when *t* is in $(-1.5 \pm 6n, 1.5 \pm 6n)$
  v(*t*) < 0 when *t* is in $(1.5 \pm 6n, 4.5 \pm 6n)$
  v(*t*) = 0 when *t* = $1.5 \pm 3n$.
- $a(t) = v'(t) = s''(t) = \frac{-\pi^2}{9}\sin\left(\frac{\pi t}{3}\right)$
- *a*(*t*) > 0 when *t* is in $(-3 \pm 6n, \pm 6n)$
  a(*t*) < 0 when *t* is in $(\pm 6n, 3 \pm 6n)$
  a(*t*) = 0 (constant speed) when $t = \pm 3n$

In this exercise, the students can use the **fMax** command. Because the curve is a parabola oriented downward, they do not need to restrict the domain.

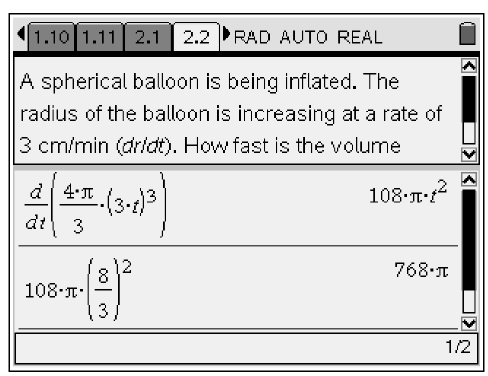


The value the students get is the time. They still have to evaluate that time in the function: s(7/2) = 196 feet.

The ball lands back on the ground when *t* = 7. Students can use **Solve** on the original equation to find that *t* = 0 when the ball was shot vertically and *t* = 7 when the ball slams into the ground.

Problem 2 – Using the chain rule in related rate problems

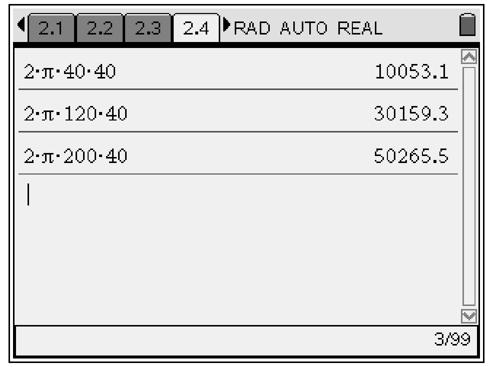
Because the rate the radius of the balloon is changing is $\frac{dr}{dt} = 3\,\text{cm/min}$, the equation for the radius at time *t* is *r* = 3*t*. Student can use this equation to find the time at which the radius of the balloon is 8 cm (*t* = 2.67 minutes).



Students will then take the derivative of the volume with respect to time. $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

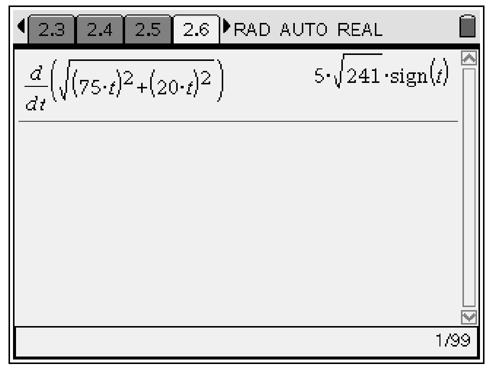
$$\frac{dV}{dt} = 4\pi(8cm)^2\left(3\,\text{cm/min}\right) = 768\pi \text{ cubic cm/min}$$

The rate that the radius of the ripple is changing is $\frac{dr}{dt} = 40\frac{\text{cm}}{\text{s}}$. So the equation of the radius at t is $r = 40t$. The area of the ripple for any radius and its time derivative is $A(t) = \pi r^2 \rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$.



At t=1, r = 40 (1) = 40 cm.
At t=3, r = 40 (3) = 120 cm.
At t=5, r = 40 (5) = 200 cm.

From the problem, the rate of the car traveling east is $x = 75t$ and the rate of the car traveling north is $y = 20t$. Because the cars are traveling at right angles, the distance between the cars is the hypotenuse of the right triangle.



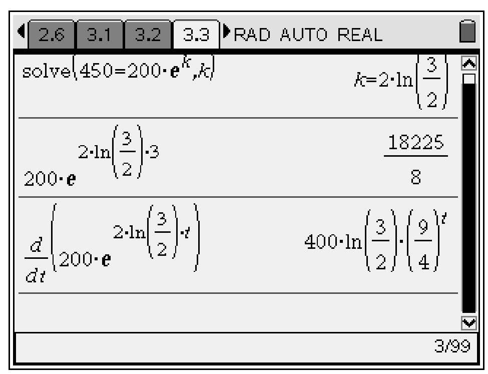
$$s(t) = \sqrt{x^2 + y^2} \rightarrow s(t) = \sqrt{(75t)^2 + (20t)^2}$$

So $s(t) = 5t\sqrt{241} \rightarrow s'(t) = 5\sqrt{241}$

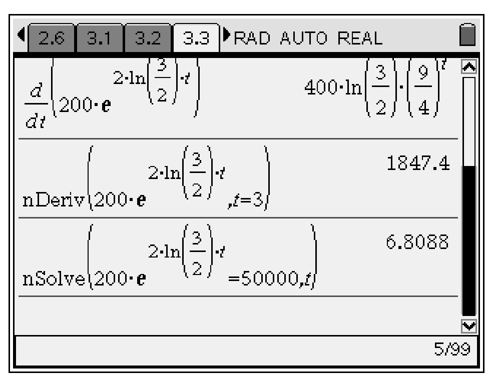
The **sign(*t*)** in the answer reflects the fact that the device does know whether t is positive or negative. Here, the answer should be positive.

Problem 3 – Growth and decay derivatives

In the growth problem,



- $A(0) = 200$
  $A(1) = 450$
  $450 = 200e^k \rightarrow 2.25 = e^k \rightarrow 2\ln(3/2) = k$
- $A(t) = 200e^{t(2\ln(3/2))}$
  $A(3) = 200e^{(3)(2\ln(3/2))} = 2278.13$
- $A'(t) = (\ln(2.25)) \cdot 200e^{t\ln(2.25)}$
  $A'(3) = (\ln(2.25)) \cdot 200e^{(3)\ln(2.25)} = 1847.4$

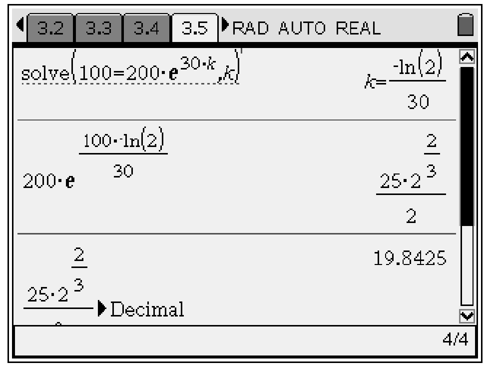


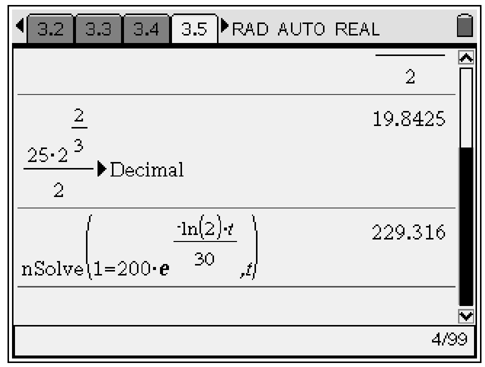
Students are to solve $A(t)$=50,000. The **nSolve** command gives the decimal time rather than a fraction. (t = 6.8088 hr)

In the decay problem,

- $A(0) = 200$,
  $A(30) = 100$ (half of $A(0)$)
  $100 = 200e^{30k} \rightarrow k = \dfrac{-\ln(2)}{30}$
- $A(t) = 200e^{-\frac{t \cdot \ln 2}{30}}$
  $A(100) = 200e^{-\frac{100 \cdot \ln 2}{30}} = 19.84$
- $A'(t) = \dfrac{-20 \cdot \ln(2) \cdot 2^{-\frac{t}{30}}}{3}$
  $A(100) = \dfrac{-20 \cdot \ln(2) \cdot 2^{-\frac{100}{30}}}{3} = -0.458$

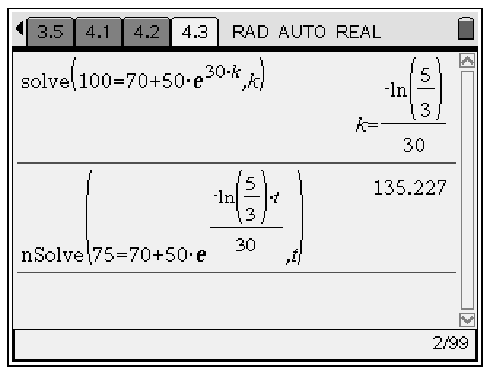
Students solve $1 = 200e^{-t\ln(2)}$ for t, **nSolve** gives 229.316 years.





Problem 4 – Cooling derivatives

In this problem, lowercase t is time and uppercase T is temperature. Remind students to be careful with the case of the letters.



- $T(0) = 120$, $T(30) = 100$, $T(sur)=70$, so $c = 50$
  $100 = 70 + 50e^{30k} \rightarrow k = \dfrac{-\ln(5/3)}{30}$.
- $T = 70 + 50e^{\frac{-\ln(5/3)}{30}t}$
- Solve for t. It will take 135.2 minutes.

Time Derivatives – ID: 9537

(Student)TI-Nspire File: *CalcAct41_TimeDeriv_EN.tns*

1.1

TIME DERIVATIVES

Calculus

Velocity and Acceleration,

Growth and Decay, Related Rates

1.2

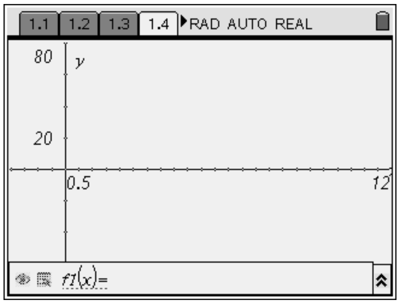
In this activity, you will explore functions with respect to time. Finding the derivatives of these functions will describe the rate of change of a variable that is dependent on time.

1.3

A roller coaster is on a launch system where the car is being pulled by the track and then is released. The function $s(t)=t^3-15t^2+48t$ is the position function where the car is being pulled by the track (0 < t < 11) and released to roll freely on the track (11 < t < 15).

Graph the function on the next page and then find the velocity and accerlation functions on page 1.5.

1.4



1.5

Find *v*(*t*) and *a*(*t*) using the **Derivative** command.

0/99

1.6

Where is the velocity postive? Negative? At rest?

Where is the accerleration postive? Negative? Constant?

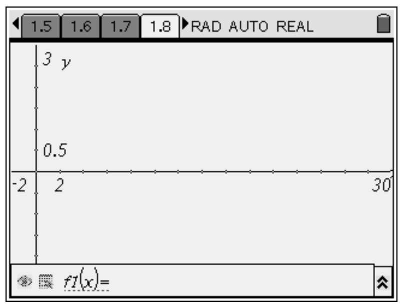
1.7

The movement of a boat sitting in an ocean can be modeled by the function $s(t)=\sin\left(\frac{\pi \cdot t}{3}\right)$.

The boat is on the top of a wave when s(t)=1 and in the trough when s(t)=−1.

Graph the function on the next page and then use page 1.9 to find the velocity and accerlation functions.

1.8



1.9

Find *v*(*t*) and *a*(*t*) using the **Derivative** command.

0/99

1.10

If a ball is shot vertically with a velocity of 112 ft/s, then its height above the ground after *t* seconds is $s(t)=16t-16t^2$. Ground is considered *s*(*t*) = 0.

What is the maximum height of the ball?

When will the ball hit the ground?

1.11

0/99

2.1

For the related rates scenarios on pages 2.2, 2.3, and 2.6, you are given the rate of change of the radius or distance with respect to time.

To solve the following problems, identify the changing quantities, write an equation that relates the two quantities, and then differentiate both sides with respect to *t*.

2.2

A spherical balloon is being inflated. The radius of the balloon is increasing at a rate of 3 cm/min (*dr*/*dt*). How fast is the volume changing when the radius is 8 cm?

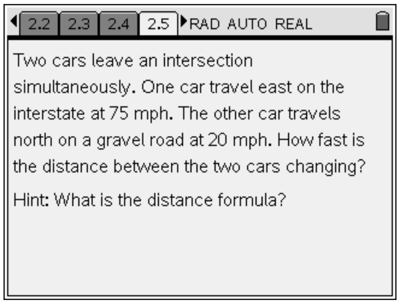
0/99

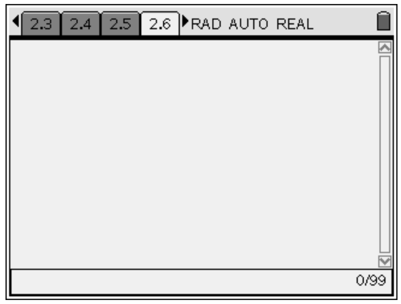
2.3

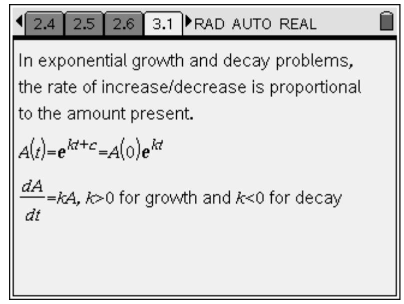
A stone is thrown into a lake, creating a circular ripple that travels outward at a speed of 40 cm/s. Use page 2.4 to find the rate at which the area of the ripple is changing when *t*=1, *t*=3, and *t*=5.

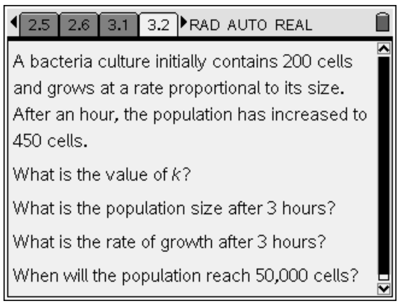
2.4

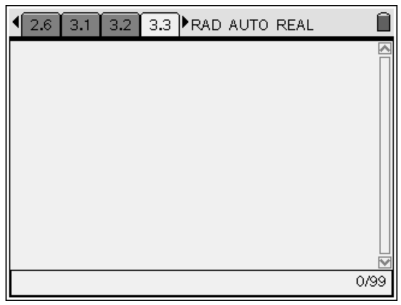
0/99

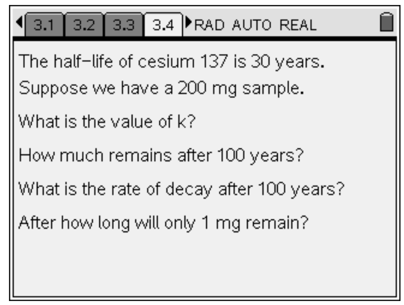

2.2 2.3 2.4 2.5 RAD AUTO REAL
Two cars leave an intersection simultaneously. One car travel east on the interstate at 75 mph. The other car travels north on a gravel road at 20 mph. How fast is the distance between the two cars changing?
Hint: What is the distance formula?

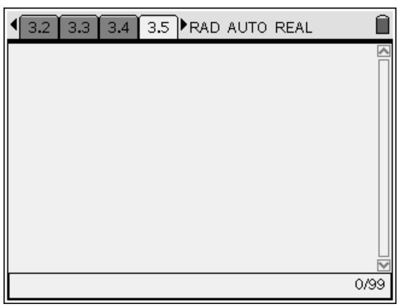

2.3 2.4 2.5 2.6 RAD AUTO REAL
0/99

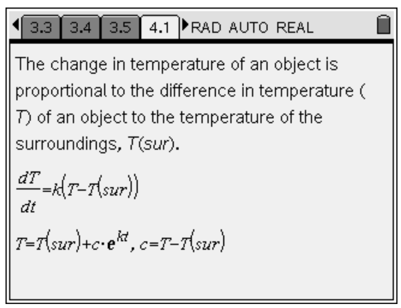

2.4 2.5 2.6 3.1 RAD AUTO REAL
In exponential growth and decay problems, the rate of increase/decrease is proportional to the amount present.
$A(t)=e^{kt+c}=A(0)e^{kt}$
$\frac{dA}{dt}=kA$, $k>0$ for growth and $k<0$ for decay

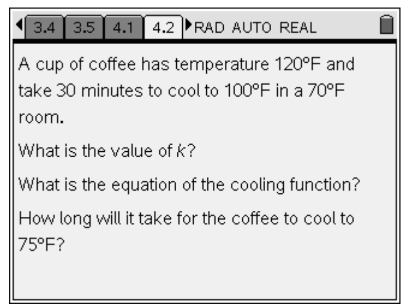

2.5 2.6 3.1 3.2 RAD AUTO REAL
A bacteria culture initially contains 200 cells and grows at a rate proportional to its size. After an hour, the population has increased to 450 cells.
What is the value of k?
What is the population size after 3 hours?
What is the rate of growth after 3 hours?
When will the population reach 50,000 cells?

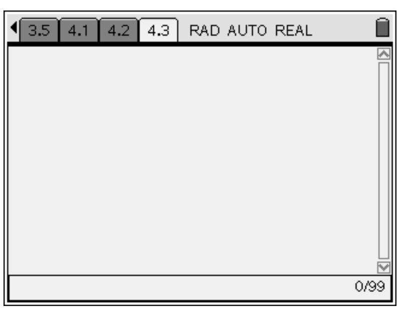

2.6 3.1 3.2 3.3 RAD AUTO REAL
0/99


3.1 3.2 3.3 3.4 RAD AUTO REAL
The half-life of cesium 137 is 30 years. Suppose we have a 200 mg sample.
What is the value of k?
How much remains after 100 years?
What is the rate of decay after 100 years?
After how long will only 1 mg remain?


3.2 3.3 3.4 3.5 RAD AUTO REAL
0/99


3.3 3.4 3.5 4.1 RAD AUTO REAL
The change in temperature of an object is proportional to the difference in temperature (T) of an object to the temperature of the surroundings, T(sur).
$\frac{dT}{dt}=k(T-T(sur))$
$T=T(sur)+c\cdot e^{kt}$, $c=T-T(sur)$


3.4 3.5 4.1 4.2 RAD AUTO REAL
A cup of coffee has temperature 120°F and take 30 minutes to cool to 100°F in a 70°F room.
What is the value of k?
What is the equation of the cooling function?
How long will it take for the coffee to cool to 75°F?


3.5 4.1 4.2 4.3 RAD AUTO REAL
0/99