EXPLORATIONS

Activity 6

Modeling Exponential Decay with a Look at Asymptotes

In the previous activity, you started your study of the exponential function, modeling exponential growth. In this activity, you will model exponential decay and learn more about asymptotes.

You should work Activity 5 before you begin.

Modeling the Experiment: Casting Out Sixes

Perform the following experiment. Before each roll, record the number of rolls that have taken place and the number of dice left.

Equipment Needed

♦ TI-83 Plus with Transformation Graphing installed.

♦ 36 number cubes (dice)

Experiment Steps

1. Count the number of dice.

2. Pick up all the remaining dice and roll them.

3. Remove any die that shows a 6 on its top face.

Number of rolls	Number of dice left
0	36
1	

Repeat the previous steps until no dice remain or you have rolled 15 times, whichever occurs first.

Sample Results:

Number of rolls	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of dice left	36	30	25	21	17	14	12	11	9	8	6	5	3	3	2	2

Use the data you collected for your analysis.

1. Enter the data in the Stat List of your TI-83 Plus. Press [STAT] and select **1:Edit**. Enter the values from the table into an empty list in the stat editor. Clear two lists if no lists are empty.

    Note: To clear a list, use the arrow keys to highlight the list name, then press [CLEAR] [ENTER].

2. Press [Y=] and clear all functions.

3. To display the plot, press [2nd] **[Stat Plot]**. Select **1:Plot1**. Turn the plot on and set the plot menu as illustrated.

    Press [ZOOM] **9:ZoomStat** to display the plot.
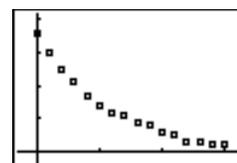
    Looking at the plot, does an exponential model appear to be reasonable? Does the problem situation lend itself to an exponential model?

    Yes, you should be taking approximately 1/6 of the dice away each time, so there seems to be a constant percent decrease. The curve should be pseudo-asymptotic to Y = 0.

4. Press [APPS] and select Transformation Graphing by pressing the number at the left of **Transfrm**. Press any key (except [2nd] or [ALPHA]) to start Transformation Graphing.

    *Note: If you do not see the screen shown at the right, select **2:Continue**.*

5. In **Func** mode, press ⌨Y= to display the **Y=** editor. Enter the general form of the exponential function, **Y1 = AB^X**. Press ⌨ALPHA **A** ⌨ALPHA **B** ⌨^ ⌨X,T,Θ,*n*.

   If the Play-Pause Mode is not selected at the left of **Y1** (>‖), press ⌨◀ until the cursor is over the symbol and then press ⌨ENTER until the correct symbol is selected.

6. Press ⌨GRAPH to re-draw the plot and the graph.

7. Select an appropriate value for **A**.

   If you started with 36 dice, then the data point (rolls = 0, dice left = 36) seems reasonable. The value of **A** in an exponential function is the value of the dependent variable, (in this case number of dice) when the independent variable is 0.
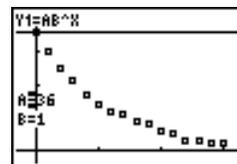
   To start with A=36, press the cursor up key ⌨▲ until **A=** is highlighted. Enter **3 6** ⌨ENTER.

   Does the y-intercept seem appropriate? If yes, continue. If no, enter a new value.

8. You could make a very reasonable estimate for **B** for this particular experiment before you start. For practice, assume that you have no idea of the value of **B**. Here you will learn to systematically try values for **B** and enter them in Transformation Graphing as a check. When the model looks reasonable, stop revising **B** and check the reasonableness of the final value.
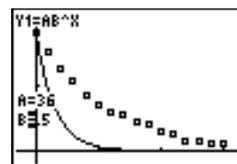
   Since the data appears to be an example of decay, **B** must be in the region 0 < B < 1. To start let **B = .5**, the middle value in the interval.

   Press ⌨▼ to highlight **B=**. Press ⌨. **5** ⌨ENTER.

9. Does your model appear to be reasonable? If yes, revise **A**. If not, test a new value for **B**. In the example, the data does not seem to be decaying as quickly as the model. Therefore, **B** must be greater than .5. **B** must be in the region .5 < B < 1. As a second guess let **B = .75**, the middle of the revised interval.
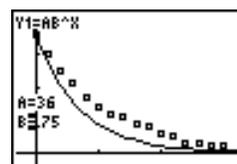
   Press ⌨. **7 5** ⌨ENTER

10. Evaluate the model again. If you think you can get a better model, continue until you feel you have as good a model as you can find. Find **B** to two decimal places.
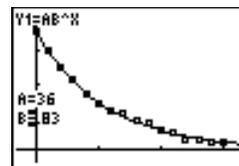
    By repeated, but systematic, guess and check, a reasonable model for the sample data would be $Y = 36(.83)^x$.

    The model has **B=.83**. Does this value make sense with respect to the problem? Explain.

    The sample data never reached zero. An exponential model never reaches 0. But the actual experiment would reach zero. How can we use this model to estimate when the dice should all be picked up?

    To determine when there should be no dice left, determine when **Y< .5**. The assumption is that as long as **Y** rounds to 1, there will be dice left. When it rounds to less than 1, we will assume no dice are left.

    Use the table feature of the calculator to determine when **Y<.5**.

11. To use the table to find **Y<.5**, press 2nd [TBLSET] to use the **TABLE SETUP** menu. Set the menu as shown at the right.

12. Press 2nd [TABLE], press ⬇ to move down until a value less than .5 appears in the **Y1** column.

    The model predicts it will take approximately 23 rolls to remove the dice.

    It is true that the exponential model never reaches zero and this situation actually does, but this model is reasonable for the situation. Some data sets are said to be *pseudo-asymptotic.* This is one of those situations.

    Note: The experiment is a discrete data set but the exponential function is a continuous model. This fact should be taken into account when the model is used.

    What does the note mean and was the model used properly?

Modeling the Experiment: Casting Out Sixes with the Special Number Cubes

Perform the following experiment. Before each roll, record the number of rolls that have taken place so far and the number of dice left.

Equipment Needed

- TI-83 Plus with Transformation Graphing installed
- 36 "special" number cubes available from your teacher

Experiment Steps

1. Count the number of dice.

2. Pick up all the remaining dice and roll them.

3. Remove any dice that show a 6 on their top face.

4. Repeat the steps until no dice remain or you have rolled 15 times, whichever occurs first.

Number of rolls	Number of dice left
0	36
1	

Sample Results:

Number of rolls	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of dice left	36	31	26	23	19	17	15	13	11	10	9	8	8	7	6	6

Use the data you collected for your analysis.

Solution

Note: Keystrokes that are the same as were needed for the first part of this activity will not be repeated.

1. Plot the data.
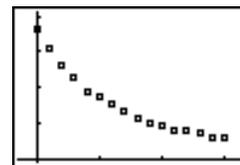
2. Does it appear that an exponential function would be a reasonable model?

   Yes, although the data does not necessarily appear to be asymptotic to **Y = 0**. Compare this plot to the data plotted in the last activity. This set appears to remain higher.

3. Activate Transformation Graphing and enter the general exponential model, **Y = AB^x**. The last values entered for **A** and **B** were from the previous dice problem. The two problems appear the same; thus these values should be reasonable.
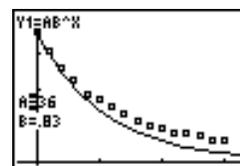
4. Does the model appear reasonable? Not really. All exponential decay curves are asymptotic, but the data and the proposed model do not appear to have the same asymptote. The asymptote for the data appears to be larger than the **Y= 0** from the previous examples.

    Since the dice were all 6-sided, the probability appears to be correct. The value **B=.83** should be reasonable.

    There are two possible methods to improve the model: try different values for **B**, or try asymptotes different from **Y = 0**.

    Observing the previous examples of exponential decay, the problem appears to be the wrong asymptote.

5. The equation for an exponential model with an asymptote other then Y = 0 is $Y = AB^X + C$ where **C** is the value of the asymptote.

    B remains the growth or decay factor.

    A is the starting point after adjustments for a non-zero asymptote. But it is no longer the y-intercept. The y-intercept is now A + C. When X = 0 the equation becomes $Y = A + C$, ($B^0 = 1$ for all values of **B**).

    For this example $A + C = 36$, since we started with 36 dice.

6. Revise your model to $Y = AB^X + C$. Try combinations of A and C where $A + C = 36$.

    A = 35   C = 1
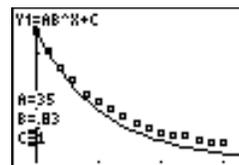
    A = 34   C = 2
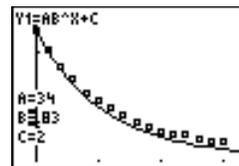
    A = 33   C = 3
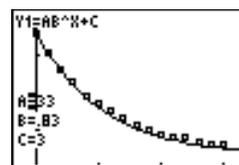
    A = 32   C = 4
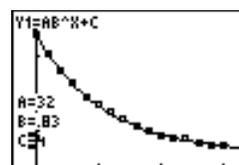
Do any of the revisions appear to depict a reasonable model? Yes, when *A = 32* and *C = 4*.

In this example, the asymptote will be *Y = 4*. This means no matter how many times the dice are rolled there will always be 4 remaining.

7. Use the table feature of the TI-83 Plus to determine when the last die containing a 6 was removed.

A Method for Visualizing the Asymptote

With the graph and stat plot showing, press 2nd [**DRAW**] **3:Horizontal**. Use the ⊡ to move the horizontal line downward with until it appears to be the asymptote. This method will give a rough idea of the location of the asymptote. You could have used this method to get a starting value for **C** when you started to develop your model.
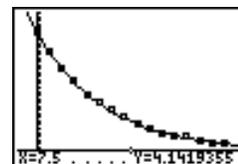
> *Note: Horizontal asymptotic behavior is an end behavior. With exponential decay it is best observed with large values of x.*

8. How do you explain the existence of a non-zero asymptote?

Homework Page

Name_____

Date _____

The data for this activity comes from the data that was collected in Activity 4. If you saved the ball bounce data in Activity 4, reload it now. If the data was not saved, redo the Ball Bounce data collection.

The data set formed by the number of the bounce and the maximum height of the ball for each bounce can be modeled with an exponential function. Trace along the data set and record the apparent maximum for each bounce. Record the points as (bounce number, bounce height).
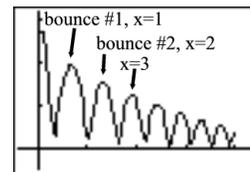
1. When tracing the data, did you find the actual maximum of each bounce?

2. What is the asymptote for the ball bounce experiment?

3. Find a model for this new data set.

Notes for Teachers

The purposes for these activities are:

♦ Re-enforce the meaning of the parameters a and b in the general exponential equation, $y = ab^x$.

♦ Introduce the non-zero asymptote form of an exponential function, $y = ab^x + c$.

♦ Give the students practice using Transformation Graphing systematically as a modeling tool.

♦ Give the students an example of a situation where a continuous model is used for a discrete experiment.

♦ Give the students a second look at asymptotes, particularly non-zero asymptotes.

The first activity is worked out almost in its entirety with sample data to give the students their first actual systematic look at modeling with Transformation Graphing. Students should repeat this experiment in small groups and go through the same steps with their own data. This will allow for a rich discussion since each group will have a different set.

The first part of this activity introduces the idea of discrete data sets being used with continuous function models. This concept is very important and should be a point of emphasis.

The sample data was very carefully collected. Students must be sure that with every roll all the dice truly roll. Also for good data, you need good dice.

This activity introduces the student to the opportunity to use modeling to try to understand a situation, not check to see if the data fits a situation.

Equipment for Experiment 2

The students will need dice without a six. The teacher decides how many dice with 6's and how many without each group gets.

♦ You can get dice like these at some joke, game, or magic stores. The best dice will have 1 - 5 with one number appearing twice.

♦ A second way to get dice like these is to make them out of cubes that can be bought at party stores or mathematics supply houses. The cubes are called number cubes and you put the numbers on yourself. If you use this method put the numbers 6 - 11 on most of the dice and 7 - 12 on a few.

♦ A third method would be to take a few of the dice and a drill and drill the two sides to look like the five side. Then do "casting out 2's."

♦ Another way is to take actual dice and put colored dots on them. Use 6 colors and tell them to pick up a die when a blue dot appears on top. Be sure not to put any blue dots on some dice and one blue dot per die on most of them.

♦ You can also make them yourself from molding you can buy at a hardware supply store.

Be sure to have your students observe all the graphs of exponential decay so they can visualize the difference between an incorrect asymptote and an inappropriate value for **b**.

In both experiments the students probably stopped before all die were removed. After they predict how many rolls would be needed to remove all the die, have them continue where they left off in the process. Remember the second experiment will not go to zero. They roll until, in our example, four dice remained.

The second activity may be done with or without the students being aware of the nature of the "special dice" set. Doing the experiment "blind" provides an opportunity for a true investigation. You might give different groups different mixes from the "special dice" set. One group might have 4 of the non-six dice, while another has 2. This will allow for different non-zero asymptotes.

CBL Practice Activity

A good CBL activity that can be used as a follow-up to this Activity is:

Experiment M5:Coffee To Go from the *CBL™ System Experiment Workbook.* The workbook was packaged with your CBL. It can also be downloaded for free from the Texas Instruments web site at **education.ti.com.**

Answers

Modeling the Experiment: Casting Out Sixes

10. **B** is the growth or decay factor. B shows what percent of the dice should remain after each roll. Since there are 6 possible faces and only 1 has a 6, the probability a die will remain is 5/6. Thus 5/6 of the dice should remain with each set of rolls. 5/6 = .83, which is the percentage that would be expected.

Modeling the Experiment: Casting Out Sixes with the Special Number Cubes

7. The last die that had a six on it would be removed, in this sample, approximately on the 23rd roll.

X	Y₁	
17	5.3473	
18	5.1183	
19	4.9282	
20	4.7704	
21	4.6394	
22	4.5307	
23	4.4405	

X=23

8. Some of the dice had no sixes. In the sample, there were 32 dice with a 6 and four dice with no 6. Therefore, four dice will never be removed.

Homework Page

1. You do not necessary have the maximum, but you have a value that is close to the maximum. Do not assume any special property for any point collected with a CBR.

2. The asymptote is 0. The ball will eventually stop bouncing. It is actually a pseudo asymptote.

3. Answers will vary depending on the data set used.