How Long

Answers

7 8 **9** **10** 11 12

TI-Nspire Investigation Student 60 min

Problem

What is the average distance between two points selected randomly within a square with side lengths of 1 unit? To solve this question using analytic methods would require mathematical techniques well beyond high school mathematics courses; however this problem can be solved approximately using simulations, the greater the number of simulations the greater the accuracy of the result. The purpose of this activity is to illustrate how solutions to such problems can be reduced into much simpler versions using simulations.

Question: 1.

Based on the information above estimate the average length of a line drawn between two random points located inside a square with side lengths 1cm.

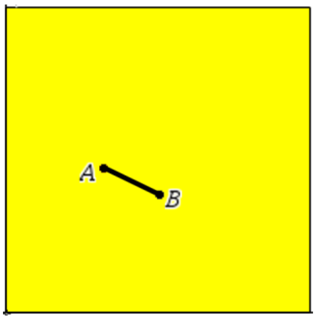
Answers will vary. Some students will predict 0.5cm, however these students are most likely thinking in just one dimension. Assuming all points inside the square are equally likely to be selected some students may average the maximum and minimum distances arriving at: $\frac{\sqrt{2}}{2}$, however, this answer is not correct either. The purpose of this question is to raise some discussion over what seems like a relatively simple question. Another interesting thing to do at this point is to take an average of the class estimates 'wisdom of the crowd'.

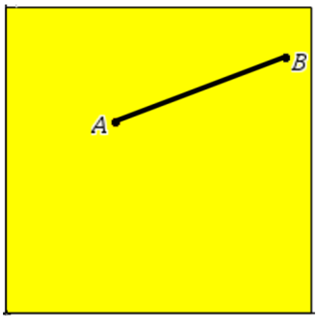
Question: 2.

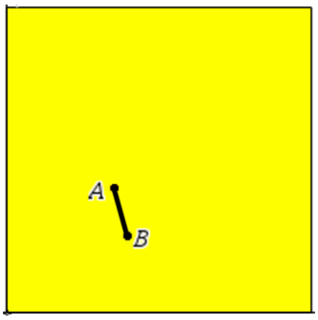
To begin here are eight samples where two points A and B have been randomly generated so that they land inside the square. The square has been scaled up to 4cm × 4cm to make it easier to measure the length of line segment AB. Measure the length of the line AB and divide it by 4 to scale it back down, record the result for each square and then compute the average length.

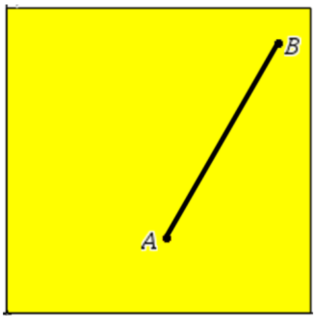
Answers will vary but should be between 0.45 and 0.49cm.

Students that write answers close to 15 have forgotten to divide by the scale factor: 4.









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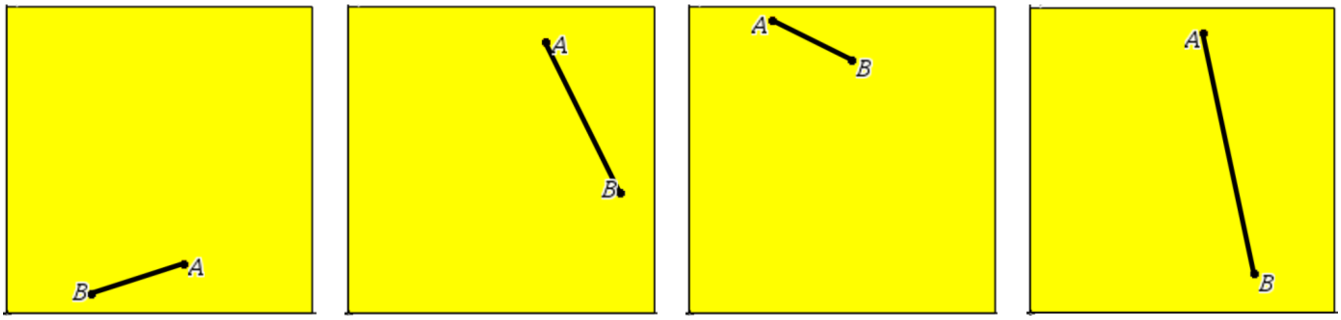


Question: 3.

The minimum length of a line segment produced using this method would be 0cm. What would be the longest possible length?

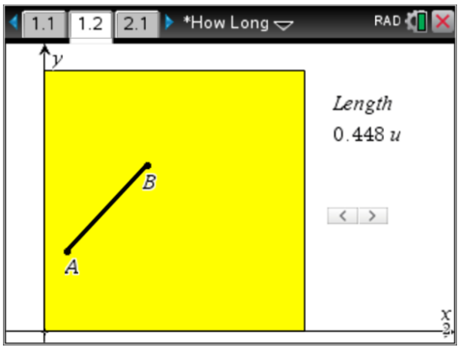
Longest length occurs when A and B are in opposite corners, using Pythagoras's theorem: $\sqrt{2}$ cm

Using a Simulator

Open the TI-Nspire file: **How Long**

Navigate to page 1.2 where a simulation of the random line segment exists.

Clicking on the spinner will generate a new line and automatically measure the length.



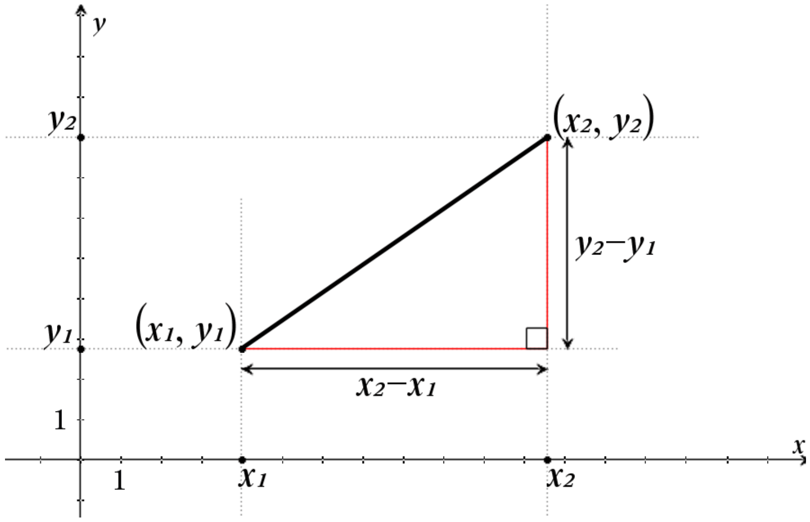
Question: 4.

Use the interactive diagram on Page 1.2 to record 20 lengths and calculate the average of these lengths.

Answers will vary: Mean length will most often be between: 0.45 and 0.59cm.

Creating Samples

The simulation on the calculator uses the Cartesian plane as a platform. Two points are plotted making it easy to produce random points. The rand() command produces a random number between 0 and 1, so two points can be generated by plotting x_1 = rand(), y_1 = rand() and x_2 = rand(), y_2 = rand(). The length of the line can then be calculated by computing the distance between the two points.



Question: 5.

Use the diagram shown opposite and Pythagoras's theorem to derive a formula for the distance between two points.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

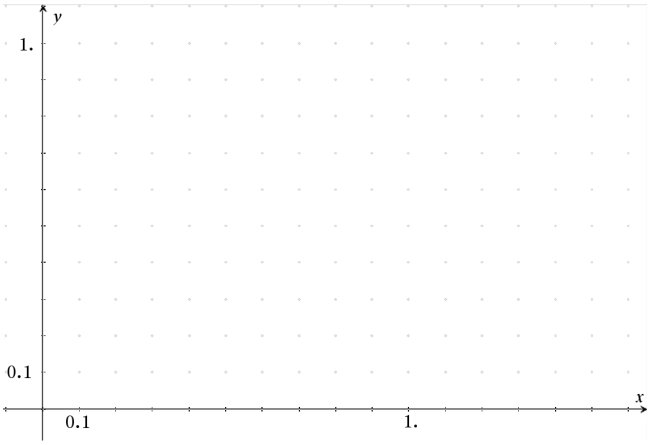
Question: 6.

Use the scratchpad to generate 4 random numbers corresponding to x_1, y_1, x_2, y_2. Record the points and plot them on the Cartesian plane.

Use your formula to calculate the distance between the two points.

Verify your answer by measuring the distance and using the scale to determine the actual distance.

Answers will vary



Creating a LARGE sample

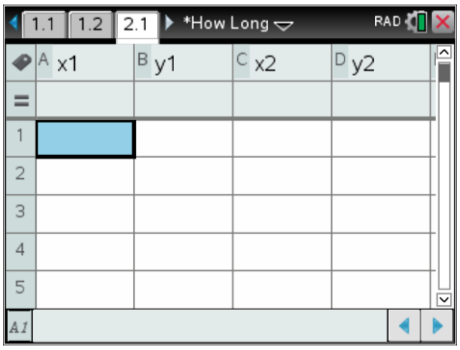
Navigate to page 2.1. A spreadsheet contains five lists:

x_1, y_1, x_2, y_2, Dist

In cell A1 type the formula:

= rand()

Place the same formula in cells: B1, C1 and D1.

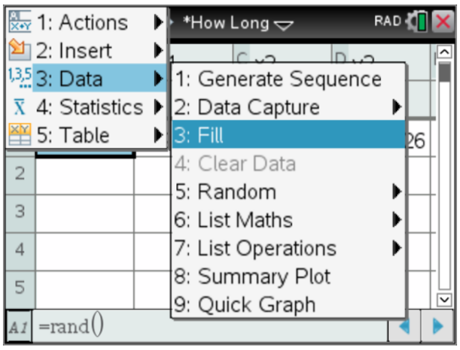


Cell formulas can be duplicated into multiple rows by 'filling' the formula down the page.

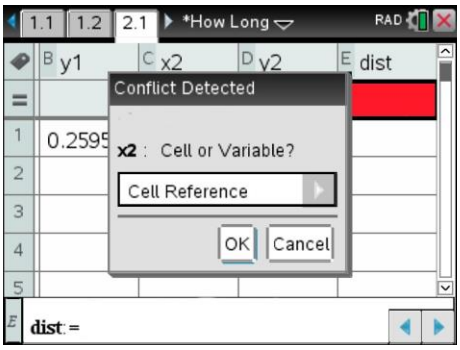
Navigate to cell A1, then from the menu select **Data** > **Fill**

Use the navigation pad (down arrow) to move down to cell A50 then press [**Enter**]. Random numbers between 0 and 1 will populate cells A1 through to A50.

Repeat this process for columns B, C and D.



Column E needs to contain the formula for the distance between two points. This formula will be placed at the top of the list in the equation bar to that all values added to the list will automatically be included and calculated. The location for the formula has been highlighted in the screen opposite. Make sure it starts with an equals '=' sign.



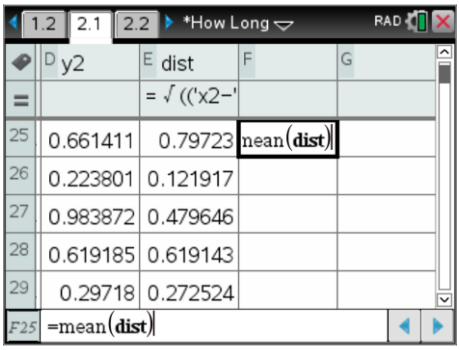
NOTE: The above image has been modified to hide the formula and highlight the location where the column formula is to be located.

Once your formula is complete you will be advised of a 'conflict'. This refers to the use of x_1, y_1 ... in your formula. The calculator needs to know if you are referring to cell locations (Cell Reference) or a variable you have created. (List name in this case.)

Select **Variable Reference** for each variable in your formula.

Navigate to cell F1. Type the formula:

=mean(dist)



Press CTRL + R to regenerate random numbers in all the cells, the calculator will automatically recompute all the distances and determine the mean. (average)

Question: 7.

Record the results of 10 sample means. Use this data to estimate the mean length of a line randomly placed inside a 1cm × 1cm square.

Answers will vary but will mostly vary between 0.45 and 0.59.

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Question: 8.

Suppose the side lengths of the square are doubled in length. What do you think the average length of a line placed inside such a square would be?

As the original square has been dilated by the same amount in all directions so too will the average length of the line, therefore the average should be double that for a 1cm × 1cm square.

Question: 9.

Adjust all relevant formulas in the spreadsheet and record the mean for 10 samples. Use this data to determine the mean length of a line randomly generated inside a 2cm x 2cm square.

Compare your results with your estimate from the previous question.

HINT: 2 × rand() will produced random numbers between 0 and 2.

Cell formulas for columns x_1, x_1, y_1 and y_2 should all read: =2rand(). No other changes are required to simulate this problem. Answers to simulations will vary.

Question: 10.

Conduct an investigation into the average length of a line placed inside a rectangle with dimensions 1cm x 2cm.

Students may change the random number formula in columns x_1 and x_1 OR y_1 and y_2 but not both.

The average length is approximately: 0.804772.

The formula for computing the average distance between two points randomly placed in a rectangle is given by: $\frac{1}{15}\left(\frac{L_w^3}{L_h^2}+\frac{L_h^3}{L_w^2}+d\left(3-\frac{L_w^2}{L_h^2}-\frac{L_h^2}{L_w^2}\right)+\frac{5}{2}\left(\frac{L_h^2}{L_w}\log\left(\frac{L_w+d}{L_h}\right)+\frac{L_w^2}{L_h}\log\left(\frac{L_h+d}{L_w}\right)\right)\right)$

where $d=\sqrt{L_w^2+L_h^2}$.

Note: On page 3.1 the above function d(length,width) has been defined. To check the average length for a 1cm x 2cm rectangle enter: d(1,2).

Extension Investigation

Suppose two random points are placed inside a box measuring 1cm × 1cm × 1cm. What would be the average length of the line connecting these two points?

- Set up the spreadsheet to produce two random points: $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$.
- Adjust your distance formula to measure the distance between two points in three dimensions.
- Collect and record a sample set of results

The average distance between two points in a 1cm × 1cm × 1cm cube is approximately: 0.66170 also referred to as Robbins Constant.[1]

Students need to add columns for z_1 and z_2 and change the distance formula to:

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ [Pythagoras's theorem in three dimensions]

[1] David P. Robbins – Robbin's is also known for his work on cyclic pentagons where each side length is a rational number (The total area is also rational) One of the unsolved problems of mathematics is to prove that the diagonals of Robbin's pentagons are always rational.

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Thinking: What would this problem look like in four dimensions? Think of the fourth dimension as time, and two objects moving around randomly in a confined space. What would be the average distance between these two objects?

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