$\lim_{x\to 0}\frac{\sin(x)}{x}$ — A Linearization Approach

by – Paul W. Gosse

Activity overview

This activity uses the linearization of the function $y=\sin(x)$ *at the point (0,0) to argue that the* $\lim_{x\to 0}\frac{\sin(x)}{x}=1$. *In particular, the linearization of* $y=\sin(x)$ *is symbolically calculated and graphed, as is the tangent line to* $y=\sin(x)$ *for values of* x, *as* $a\to 0^{+}$ *(which is done by means of a slider). This is a natural and strong approach for developing an understanding of* $\lim_{x\to 0}\frac{\sin(x)}{x}$.

Traditionally, this limit would be approached from a table of values point of view, or a geometric point of view which is actually similar to the approach offered here, or, later, using L'Hospital's Rule. What is different about this approach is that the limit $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ *is approachable in terms that seem much more concrete to the average Calculus student, and de-mystifies* $\lim_{x\to 0}\frac{\sin(ax)}{ax}=1$ *quite quickly. In particular, visualization of the tangent/linearization at a point, is dynamic and interactive producing a convincing argument that* $\sin(x)=x$ *as* $x\to 0^{+}$.

Concepts

First derivative, chain rule, tangent line, linearization of functions, limits, fundamental trigonometric limit, L'Hospital's Rule.

Teacher preparation

This activity is fairly robust and requires little support in terms of preparation. Prior to completing this activity, students should be able to determine the linearization of $y=\sin(x)$ *at a point, and should understand the Chain Rule since* $\lim_{x\to 0}\frac{\sin(ax)}{ax}=1$ *engages the chain rule to justify the linearization. The activity uses the derivative command* $\frac{d}{dx}\left(f(x)\right)$ *found in the Calculations-Calculus menu and skills common to determining the tangent line to a curve at a given point. This activity may best be done as a prelude to L'Hospital's Rule after establishing the derivative of* $y=\sin(x)$.

Classroom management tips

Few classroom tips are required. The activity is self-explanatory. Students should compare this result to L'Hospital's Rule and consider why, as $x\to a$, *this approach can be effective.*

TI-Nspire Applications

Notes (including Q&A), Calculator, G&G.

$\lim_{x \to 0} \frac{\sin(x)}{x}$ — A Linearization Approach

by: Paul W. Gosse
Grade level: secondary, college
Subject: mathematics
Time required: 45 to 90 minutes

Materials: TI-Nspire CAS

Step-by-step directions

The activity is self-explanatory with very little preparation and explanation needed. Answers, however, are offered below.

Assessment and evaluation

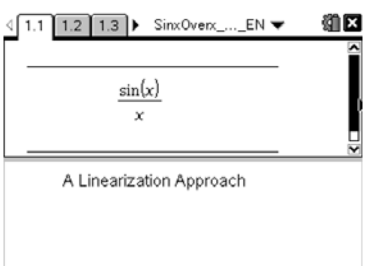
- *The activity may be assessed by asking students to evaluate similar trigonometric limits as* $x \to 0$ *justifying their results.*
- *Answers: [1.6] Notice that the tangent line approaches* $y = x$ *; [1.8] The equation is* $y = x$ *once* $x = 0$ *; [1.10] This page links pages 1.6 & 1.8 together as the same equation; [1.11] The limit may be re-written as* $\lim_{x \to 0} \frac{x}{x}$ *which simplifies to* $\lim_{x \to 0} \frac{1}{1} = 1$*; [1.15]* $\lim_{x \to 0} \frac{\sin(5x)}{5x} = \lim_{x \to 0} \frac{5x}{5x} = \lim_{x \to 0} \frac{1}{1} = 1$*; Similarly, [1.16]* $\lim_{x \to 0} \frac{\sin(ax)}{ax} = \lim_{x \to 0} \frac{ax}{ax} = \lim_{x \to 0} \frac{1}{1} = 1$.

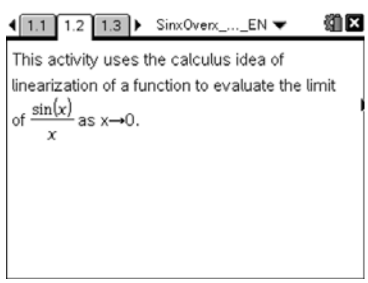
Activity extensions

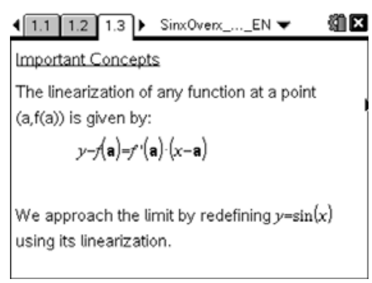
- *Students could apply the same approach to* $\lim_{x \to 0} \frac{\cos(x)}{x}$ *or similar derivatives.*
- *In many cases, determining the limit of the ratio of two functions can be clarified by examining the ratio of their respective linearizations.*

Student TI-Nspire Document

SinxOverx_ALinearizationApproach_EN.tns

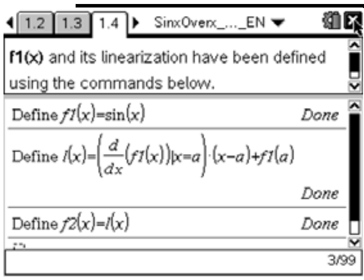


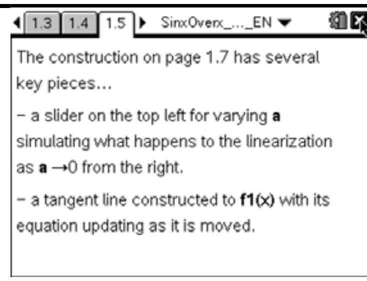


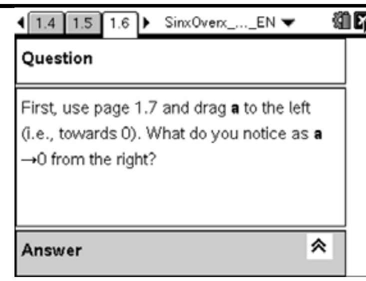


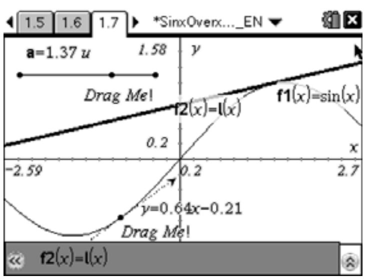
by: Paul W. Gosse Grade level: secondary, college
Subject: mathematics
Time required: 45 to 90 minutes

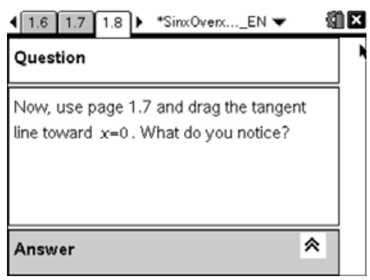
Materials: TI-Nspire CAS

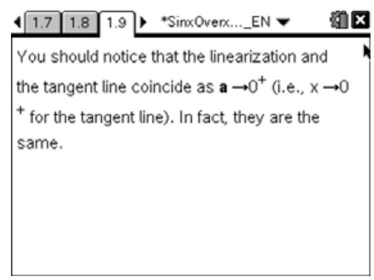


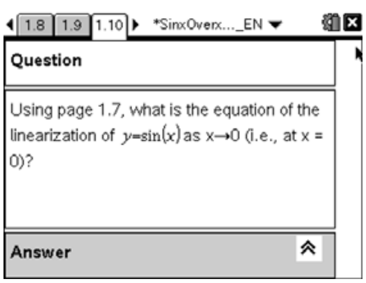


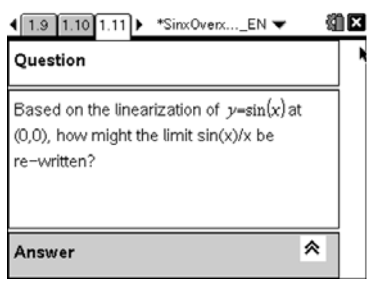


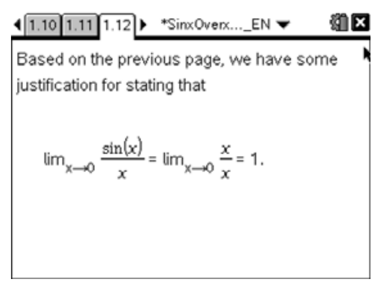


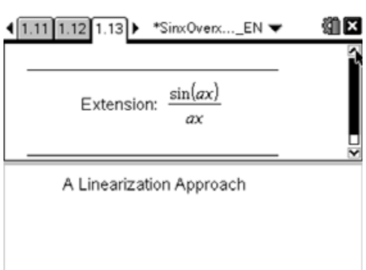


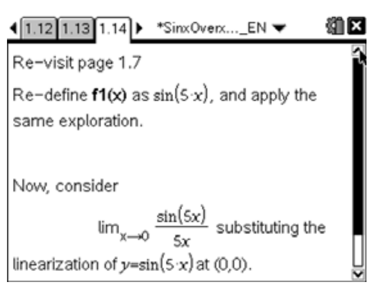


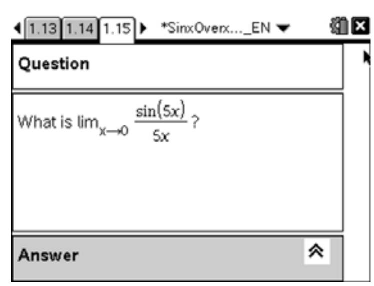


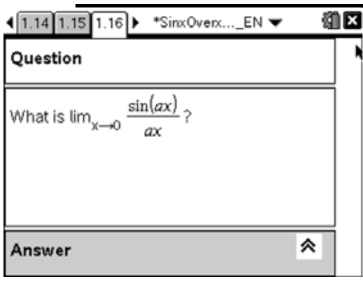


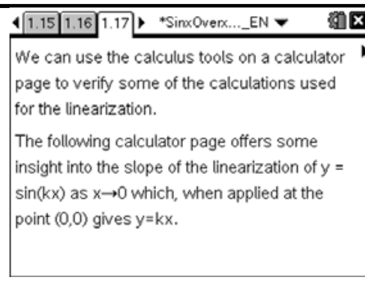


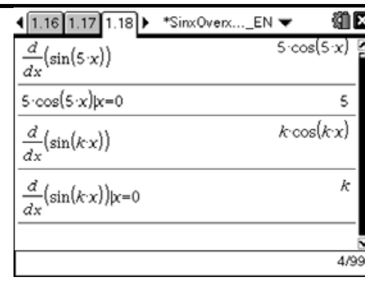


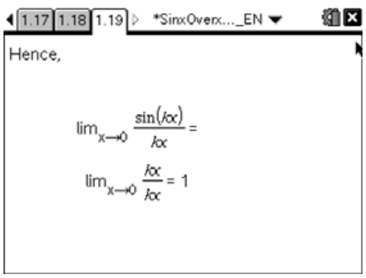





1.14 1.15 1.16 *SinxOverx..._EN
Question
What is $\lim_{x \to 0} \frac{\sin(ax)}{ax}$?
Answer


1.15 1.16 1.17 *SinxOverx..._EN
We can use the calculus tools on a calculator page to verify some of the calculations used for the linearization.
The following calculator page offers some insight into the slope of the linearization of y = sin(kx) as x→0 which, when applied at the point (0,0) gives y=kx.


1.16 1.17 1.18 *SinxOverx..._EN
$\frac{d}{dx}(\sin(5\cdot x))$ $5\cdot\cos(5\cdot x)$
$5\cdot\cos(5\cdot x)|x=0$ 5
$\frac{d}{dx}(\sin(k\cdot x))$ $k\cdot\cos(k\cdot x)$
$\frac{d}{dx}(\sin(k\cdot x))|x=0$ k
4/99


1.17 1.18 1.19 *SinxOverx..._EN
Hence,
$\lim_{x \to 0} \frac{\sin(kx)}{kx} =$
$\lim_{x \to 0} \frac{kx}{kx} = 1$