Kepler's Third Law – ID: 8515

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Time required
2 hours

Activity Overview

Johannes Kepler (1571-1630) is remembered for laying firm mathematical foundations for modern astronomy through his study of the motions of the planets. Although nearly blind himself, Kepler used the precise planetary data of astronomer Tycho Brahe and discovered the mathematics behind these thousands of observations, collected over many years. Kepler's first and second laws describe the elliptical paths and the velocity of the planets as they move around the sun.

Kepler's Third Law compares the motion of the various planets. Each of them is characterized by two quantities in particular:

- the mean distance from the sun
- the time of revolution around the sun

Are these two quantities related?

The aims of this activity are:

- to establish whether a power function is a good fit for the data
- to discover the power function that "best" matches the data

Background

The historical background is the Scientific Revolution of the 1600s. Kepler's merit was that he believed there was a regular function that would fit the observed data. Subsequent work by Newton proved that the law linking the planets' time of revolution *T* to their distance from the sun *R* was:

$$R^3 = \frac{GM}{4\pi^2}T^2,$$

where *G* is the universal gravitational constant (approximately $6.67 \cdot 10^{-11}$ Nm^2/kg^2) and *M* is the sun's mass (approximately $2 \cdot 10^{24}$ kg).

Concepts

Regression line, the least squares method, power functions, logarithmic properties

Teacher preparation

This activity requires prior knowledge of the notion of "regression lines", which may have been covered when dealing with quadratic functions. For one approach to introducing this important concept, refer to the Appendix as the end of this activity.

Classroom management tips

Students should work in small groups on the initial examination of the problem. This stage is useful for teasing out all its implications. It is also a good idea for the teacher to go back to the students' approaches at the end, comparing them with the solution found.

Students' prerequisites

- Linear functions
- Quadratic functions
- Power functions
- Basic knowledge of the **Calculator**, **Graphs & Geometry**, **Lists & Spreadsheet** applications

Step-by-step directions

1. The activity begins with the following table showing, for the planets visible to the naked eye, the mean distance *r* (measured in 10^9 m) and the time of revolution *t* (measured in 10^6 s).

A	B r	C t	D
Mercury	58	8	
Venus	108	19	
Earth	150	32	
Mars	228	59	
Jupiter	778	375	
Saturn	1423	929	

From the table, we may observe that *t* increases with *r*: More distant planets require longer to revolve around the sun. What type of increase is it? For example, is it linear? The answer is not obvious just from studying the table. For a better understanding of the problem, let us plot a graph with the points.

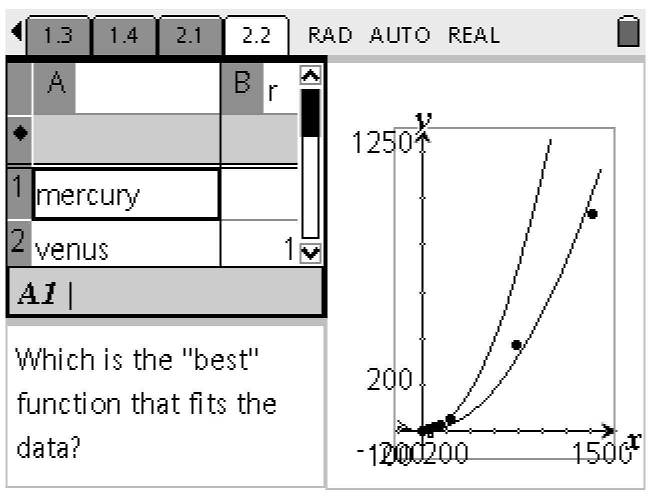
The graph shows a regular pattern but not a linear one: time increases more quickly than distance. Whatever curve fits the data, it appears to be “more than linear”; it also passes through the origin (0,0).

We can now give the students (purely at an experimental level for the time being) the problem of selecting a function, and observe and guide their approaches.

For instance, the fact that the desired curve appears to pass through the origin argues against the choice of an exponential function, of the type $\boldsymbol{y = ab^x}$.

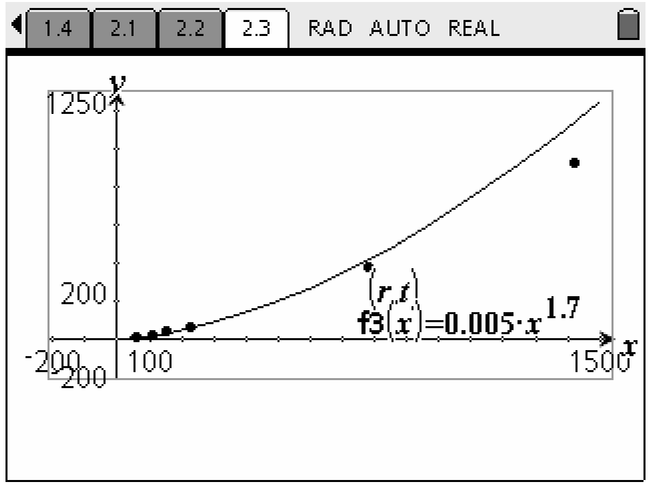
A reasonable choice, which students might suggest, is the function $\boldsymbol{y = ax^2}$.

Let us try defining the function $\boldsymbol{f1(x)=x^2}$ and changing its amplitude *a* with the cursor: we can see that while the curve fits the points for the planets closest to the sun, it is quite a long way from those for Jupiter and Saturn. Also, we cannot approximate these last two points in any convincing way.



2. Based on the analyses and discussion with the students, and on the observations we have made so far, we can suggest that a good model for the data may be a **power function**, i.e. a function of the type $\boldsymbol{y = ax^b}$ where $\boldsymbol{b > 1}$.

   We may then ask students to work in small groups and look experimentally for a power function (and therefore look for the parameters *a* and *b*) that fits the observed data as closely as possible. The following graph shows the power function when $a = 0.005$ and $b = 1.7$.



The teacher thus has material for discussion at the end of the activity.

3. Let us now provide convincing proof that the power function is a good model. If x and y were linked by a relationship of the type $\boldsymbol{y = ax^b}$ then, taking logarithms, we would have

$$\ln(y) = \ln(a) + b\ln(x)$$

In other words, a **linear** relationship between $\boldsymbol{X=\ln(x)}$ and $\boldsymbol{Y=\ln(y)}$ with slope b and intercept ln(a):

$$Y = bX + \ln(a).$$

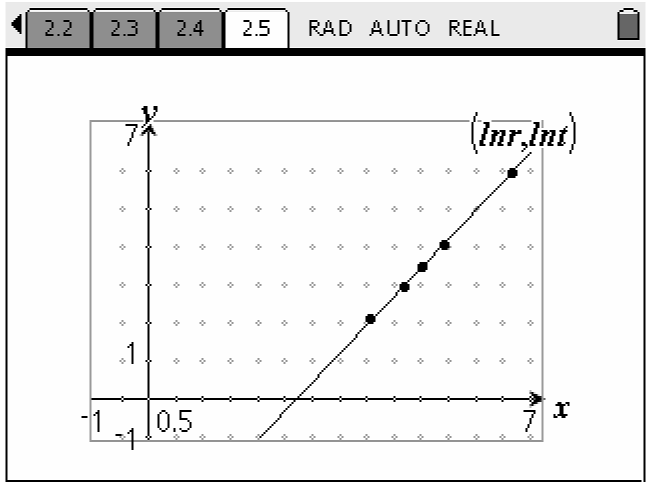
We now go back to the table and add columns **D** and **E**, in which we calculate ln(r) and ln(t) respectively.

2.2 2.3 2.4 2.5 RAD AUTO REAL

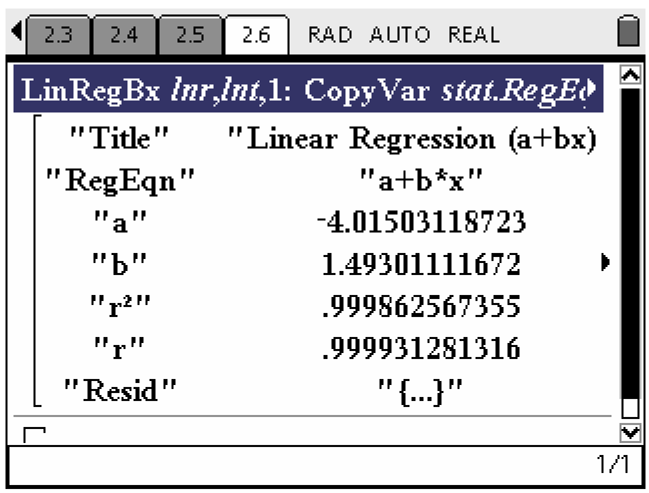
	A	B r	C t	D lnr	E lnt
♦				=ln('r)	=ln('t)
1	mercury	58.	8.	4.0604...	2.0794...
2	venus	108.	19.	4.6821...	2.9444...
3	earth	150.	32.	5.0106...	3.4657...
4	mars	228.	59.	5.4293...	4.0775...
5	jupiter	778.	375.	6.6567...	5.9269...

A1 | mercury

Graphing these points, with the logarithm of r on the horizontal axis and the logarithm of t on the vertical axis, clearly shows a linear progression.

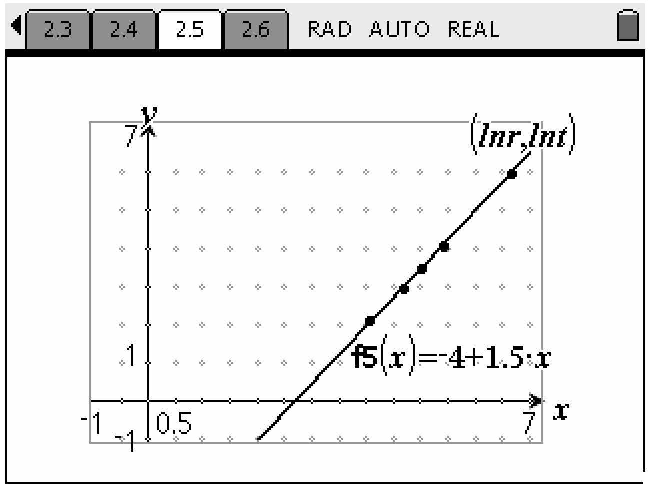


4. We now have to calculate the linear function that “best” fits these data, using the least squares method. In **Calculator** mode, we define *r1*=ln(*r*) and *t1*=ln(*t*) and calculate the regression line.



The line which best fits the linear progression between ln(*r*) and ln(*t*) is therefore approximately

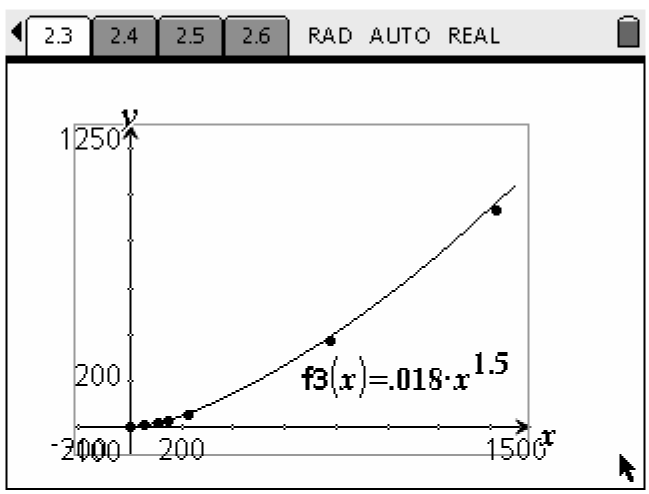
$$Y = 1.5X - 4$$



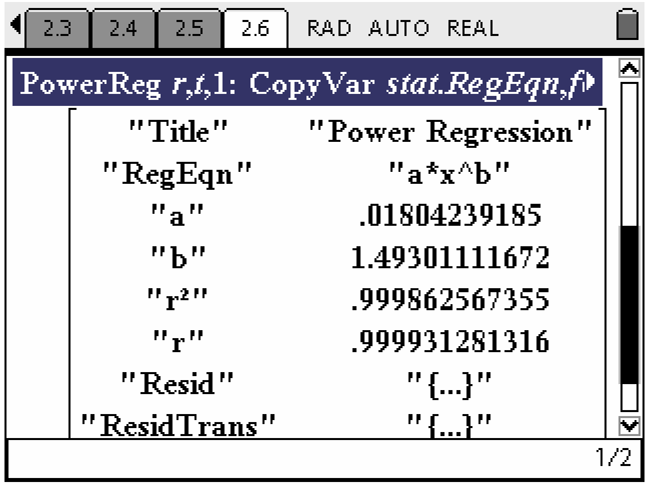
Since **ln(*a*) = –4**, then ***a* = exp(–4) ≈ 0.018**, and ***b* = 1.5**.

The power function we are seeking is approximately

$$\boldsymbol{y = 0.018 \cdot 1.5^{x}}$$



The graph for this power function is a good fit for the observed data. Moreover, if we calculated the power function directly using the least squares method (via the *PowerReg* command in **Calculator**) we would obtain the same result.



5. We can now compare the students' approaches in stage 2 with the solution identified; we can see who got closest to the solution by calculating

$$T(a,b) = \sum_{i=1}^{n}\left(ax_i^b - y_i\right)^2 \;;$$

The person closest to the solution (according to the least squares method) is the one with the lowest value for *S.*

For the teacher. N.B.: the values *a* and *b* that minimize *T*(*a*,*b*) are not the same as those that minimize

$$S(a,b) = \sum_{i=1}^{n}\left(\ln(a) + b\ln(x_i) - \ln(y_i)\right)^2 \;;$$

however, in general, they represent a good approximation.

Minimizing *T*(*a*,*b*) implies solving a non-linear system of equations which, in general, does not allow for a symbolic solution; however, by using the following substitutions

$X = \ln(x)$

$Y = \ln(y)$

$A = b$

$B = \ln(a)$

we obtain the linear function

$Y = AX+B$

S(*a*,*b*) is therefore linear, and to minimize *S* we can use a simple property of quadratic functions, as we have seen.

Assessment and evaluation

Possible questions may include:

- Calculate the power function which (according to the least squares method) “best” fits the points (2,1), (4,2), (5,4).
- Establish which of the power functions $\boldsymbol{y=0.3x^{1.4}}$, $\boldsymbol{y=0.4x^{1.5}}$ best fits the points (2,1), (4,2), (5,4).
- An asteroid revolves around the sun at a mean distance of 500 million km. What is its time of orbit?
- Uranus takes approximately 30,700 days for one complete orbit of the sun. What is its mean distance from the sun?

Activity extensions

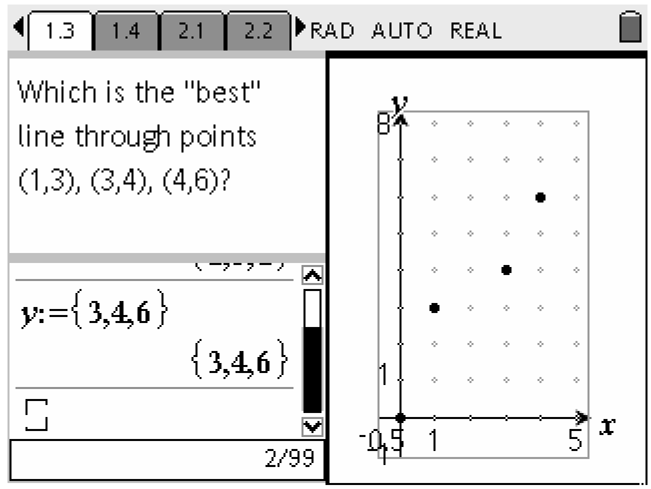
- The most obvious next step is to move on to the exponential regression function, i.e. for a given n points, calculate the function of the **type** $\boldsymbol{y = ab^x}$ which fits them best.

  Using a method similar to that for the power function, by taking logarithms we obtain

  $\boldsymbol{\ln(y) = \ln(a) + x\ln(b)}$**;** in other words, two quantities x and y are in an exponential relationship if x and $\ln(y)$ are in a linear relationship, with slope $\ln(b)$ and intercept $\ln(a)$.

APPENDIX: A TI-Nspire CAS Introduction to Least Squares Regression

For one approach, students could be given the following problem, with no preparation:



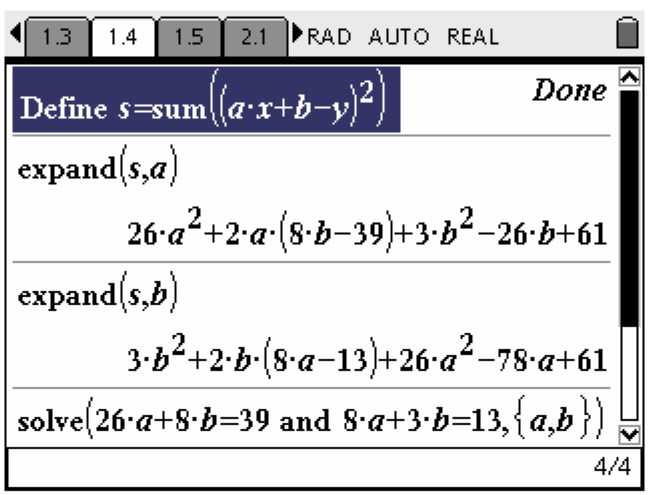
Find the "best-fit line" y=ax+b which passes through the points (1,3), (3,4), (4,6).

If the "best-fit line" is defined as the one that minimizes the sum of the squares of the differences between the "observed" ordinate y_i and the "theoretical" ordinate ax_i+b-y_i

$S(a,b) = \sum_{i=1}^{3}\left(ax_i + b - y_i\right)^2 =$

$(a+b-3)^2+(3a+b-4)^2+(4a+b-6)^2$

we can easily use the **CAS Calculator** Application to calculate the parameters *a* and *b* of the regression line.



S(a,b) is actually a quadratic function of *a* and *b*. If we consider *b* to be fixed, then *S(a,b)* is a quadratic function of *a*, with a positive coefficient whose minimum value corresponds to the point:

$$\boldsymbol{a = (39-8b)/26}$$

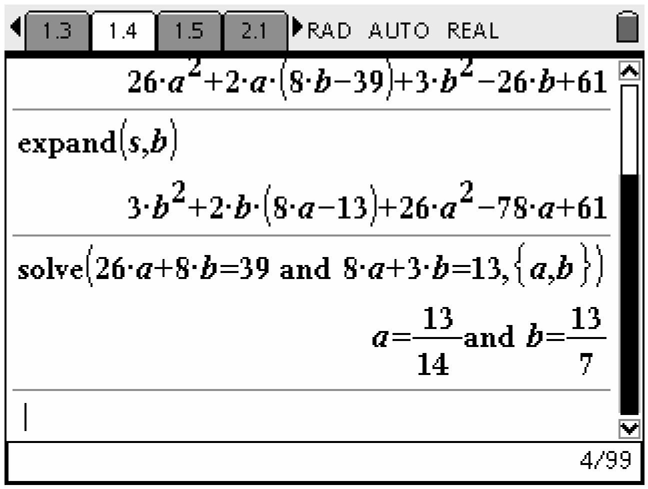
Similarly, if we consider *a* to be fixed, we obtain a quadratic function of *b* whose minimum value is:

$$\boldsymbol{b = (13-8a)/3}$$

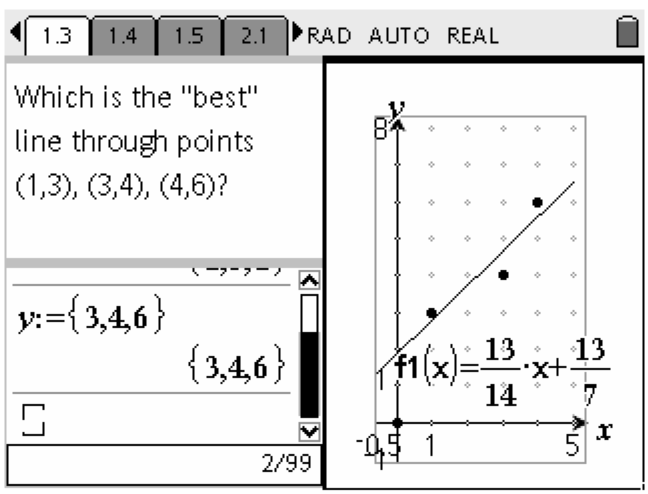
We may then solve the simultaneous equations

26*a*+8*b* = 39

8*a*+3*b* = 13



The desired regression line is therefore ***y* = 13/14*x*+13/7**.



We can now go back to the approaches developed by the students.