## Math Objectives

- Students will be able to interpret variability in the sampling distribution of a sample statistic.
- Students will evaluate the usefulness of a statistic by studying its sampling distribution.
- Students will be able to identify a biased estimator that results in predominantly over or under estimates of the population parameter.
- Students will make sense of problems and persevere in solving them (CCSS Mathematical Practices).
- Students will model with mathematics (CCSS Mathematical Practices).

Vocabulary

- bias
- dotplot
- mean
- median
- maximum
- minimum
- parameter
- statistic
- variability

About the Lesson

- This lesson involves estimating the largest value from a population of consecutive numbers starting at 1.
- As a result, students will:
    - Create rules to estimate the largest number of a population based on random samples from the population.
    - Analyze rules using graphs of the sampling distributions of a sample statistic.
    - Analyze rules to determine an estimate of the largest value from a population.
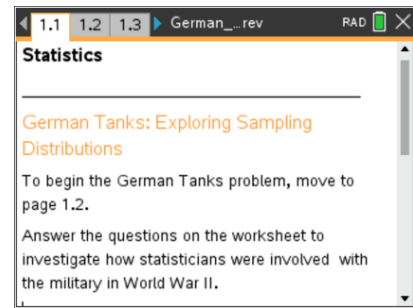
TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing `ctrl` `G`.

Lesson Files:
Student Activity
German_Tanks_Exploring_Sampling_Distributions_Student.pdf
German_Tanks_Exploring_Sampling_Distributions_Student.doc

TI-Nspire document
German_Tanks_Exploring_Sampling_Distributions.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

Teacher Tip: The German tank problem is a "classic problem." A search on the web will reveal many sites, books, and statistical notes addressing this problem and its unique place in history. Consider sharing these resources with your students for expanding the discussion. Be advised, however, that many of these sites include the method derived by the statisticians that is part of this activity. You might consider discussing the links at the conclusion of this activity so that students understand the key features that make a "good rule." Allow students the opportunity to discover rules by using the technology features in the .tns file.

Move to page 1.1.

Similar to the challenge faced by statisticians during World War II, you will be provided a sample of numbers representing serial numbers of German war tanks. Assume that the serial numbers were consecutively numbered from 1 to the number of the last manufactured tank. Your goal is to determine the number of manufactured tanks, that is, the largest serial number.
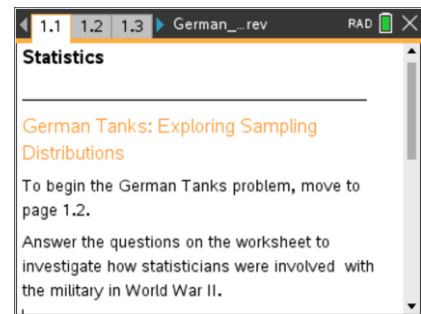
Move to page 1.2.

Follow the directions on Pages 1.2 and 1.3 to seed the random number generator. This will result in different random samples for each member of the class.

Move to page 1.4.

Click the right arrow on Page 1.4 to generate a random sample of five serial numbers in the column labeled **capture** and to see a dotplot of the sample. These values represent your sample of captured serial numbers.
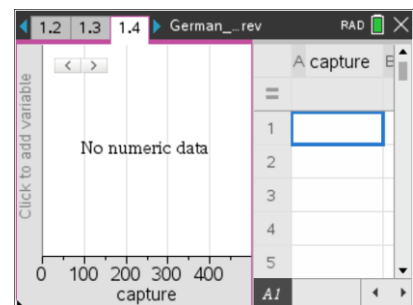
1. a. The smallest serial number would be 1. Did you capture that serial number? The largest serial number is not known. Do you think you captured it? Why or why not?

   Sample Answers: It is not likely that tank 1 has been captured. It is also not likely that the largest value has been captured.

b. The goal of this investigation is to create a *rule* to estimate the largest serial number from the population of tanks. Before you compare samples, think about how you might use the serial numbers you have to make an estimate of the number of tanks. Describe in words the rule you would apply to your sample to estimate the number of tanks.

Sample Answers: Student answers will vary. For example, I found the mean of the sample of the captured serial numbers, and multiplied the mean by 2.

> **Teacher Tip:** Students typically generate rules involving the mean, the median, or some combination. Encourage your students to generate their own rules. Help them apply their rules in the following pages. Some students might require a little more direction to understand what they are being asked to do. As a first step, encourage students to place the serial numbers in order. Ask them how this organized look at the sample might indicate an estimate to the largest value. The example in the following paragraphs provides an opportunity to discuss one of several reasonable rules. Use this example only if a student is unable to generate his or her own rule. Explain that there are several possible rules that could generate estimates to our question and that it is ok to just make a rule up and see what happens.

c. Based on your sample of five numbers, create a formula to estimate the largest serial number that represents the rule you described above.

Sample Answers: The formula for my rule is 2 times mean of the captured sample.

Move to page 1.5.

Page 1.5 is designed for you to enter your rule into cell A1 of the spreadsheet.

2. Move the cursor over cell A1 and click. A1 will now be highlighted. Type an "=" into cell A1. The equal sign indicates that a formula (or rule) will follow. Type in your rule, move the cursor to a position outside the parentheses, and press ⏎ enter ⏎. The estimate of the largest value based on your rule will be displayed.
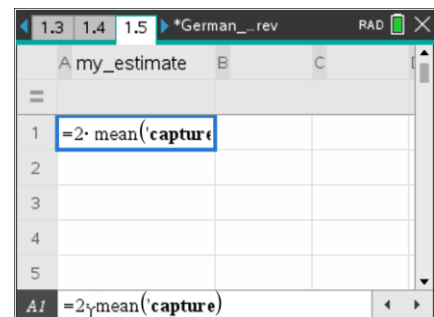
- As an example, consider the rule: "Twice the value of the mean of the sample." This rule would require the following formula to be entered:

$$= 2 \cdot \text{mean(capture)}$$

Note: It is important to refer to the list **capture** when you want values such as the mean, median, max, min, or sum of the sample.

> **Teacher Tip:** The directions above assume students and teacher are somewhat familiar with entering a formula into a cell of a TI-Nspire spreadsheet. The following expressions might be used as a formula for generating and capturing data into the *capture* column:
> max(*capture*)
> min(*capture*)
> median(*capture*)
> mean(*capture*)

Move to page 1.6.

3. Examine Page 1.6.
   a. What does the **capture** dotplot represent?

   Sample Answers: The **capture** dotplot is another sample of five serial numbers randomly selected from the set of possible serial numbers.

   b. What does the **my_estimates** dotplot represent?

   Sample Answers: The dotplot of *my_estimates* is the estimate of the largest number of the population of German tanks based on the rule I entered on page 1.5.

4. Share your estimate of the number of tanks with other students. Were any of the estimates very different from yours? What might explain the difference?

   Sample Answers: The rules used by other students are different from my rule. In addition, each of the samples is different. My rule used the mean of the captured numbers. As the samples are different, the mean will be different.
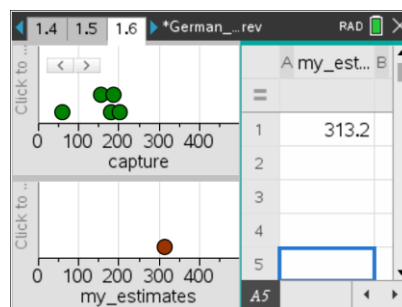
> **Teacher Tip:** Students might observe that their estimate of the largest number is less than the largest number of some of the samples, possibly even their own sample. Students should continue the activity without changing their rules. The goal of this activity is to investigate properties of the rules applied to the captured sample and not necessarily to nail down one specific estimate of the number of tanks.

Suppose you had a different sample of five serial numbers.

5. Use the right arrow to generate a new sample, assuming all of the tanks were back in use by the German army when this new sample of five tanks was captured. Determine the estimate of the number of tanks from this new sample using your rule. Compare your first and second estimates.
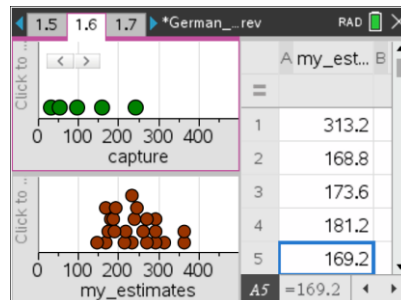
<u>**Sample Answers:**</u> Answers will vary based on the samples and rules. Using the rule on my first sample, I estimated the number of tanks to be 236. From my second sample, I estimated the number of tanks to be 175.

It might be useful to know how your rule will work for different samples.

6. a. Use the right arrow to generate estimates for at least 30 samples.

   b. Examine the dotplot of **my_estimates.** Describe your rule and the distribution of estimates for the total number of tanks using your rule.

<u>**Sample Answers:**</u> My rule was twice the mean. My estimates appear to pile on top of each other around the value of 270 and the distribution is fairly symmetric. I had an estimate as high of 370 tanks and one as low as 100 tanks.

> **Teacher Tip:** Student descriptions should include the shape, center, and spread of the distribution. Some rules might appear to generate a wide spread of values with a cluster of estimates around a certain value.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 1 at the end of this lesson.

Move to page 1.7.

7. The spreadsheet displayed on Page 1.7 contains the estimates for the total number of tanks using four separate rules. The estimates derived from your rule are in Column D and are labeled **my_estimates**. Click in the cell at the top of each column. Describe the rule that generates the estimates in that column.

	A ma...	B twi...	C mul...	D my_...	E
=					
1	206.	362.	241.2	313.2	
2	239.	110.	280.8	168.8	
3	168.	104.	195.6	173.6	
4	142.	168.	164.4	181.2	
5	183.	126.	213.6	169.2	

Sample Answers:
Column A: the rule is the maximum value of the captured sample plus 5.
Column B: the rule is twice the median of the captured sample.
Column C: The title for column C suggests that this rule also uses the maximum value of the captured value, but I am not sure what the rule really is.

> **Teacher Tip:** "multimaxs" represents the rule used by the statisticians in the analysis of the German tanks. Students will not be able to determine from the title or looking at the values the specifics of this rule. Allow this rule to be a mystery until the end of the activity.

Move to page 1.8.

8. Four dotplots are displayed on Page 1.8. Each dotplot displays the estimates derived from the four rules you saw on Page 1.7.
   a. The vertical line represents the actual number of tanks in the population for a particular month. Click on the line. How many tanks does this simulation assume the Germans manufactured this month?
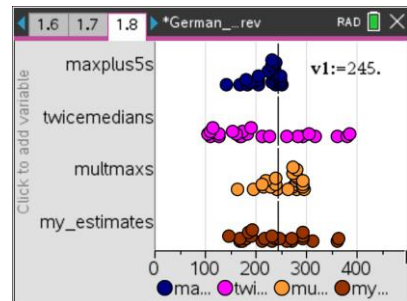
Answer: 245 tanks.

b. Hover over each of the dotplots. Record the minimum and maximum of the distribution for each rule in the table below. (Note that clicking in a white space will deselect a point.)

Sample Answers:.

Rule	Minimum value of sample	Maximum value of sample
maxplus5s	94	250
twicemedians	54	448
multimaxs	106	193
my_estimates	93	383

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 2 at the end of this lesson.

Teacher Tip: Students might be surprised that the distributions for the three common rules will be similar. The rule for **maxplus5** suggests that the estimates are almost always too low; **twicemedian** seems to have a very large spread. The range of values for column **multimaxs** is generally smaller than the ranges noted for the other rules, and the distribution seems to be centered at the actual number of tanks.

An estimate generated by a rule or formula from a sample is considered a **statistic**. A rule that consistently generates statistics less than the true value or consistently greater than the true value is an example of **bias**. One outcome of bias would be a dotplot in which most of the dots are located less than the true value. This dotplot indicates that estimates from this rule are consistently less than the true value. Similarly, a dotplot what would have most of the dots located greater than the true value would be another example of bias. Biased rules should be eliminated from consideration when looking for the best estimate.

9. a. Do any of the dotplots suggest the statistic is biased?

Sample Answers: For the rule **maxplus5s**, nearly all of the estimates are less than the 245, the actual number of tanks.

b. Why do you think this rule generates biased results?

Sample Answers: The maximum from a sample is generally not the largest number of the population. Adding 5 to whatever is the largest of the sample will generally not exceed the largest number of the population. For this rule to produce a value greater than the largest value, the maximum value of the sample would need to be very close to the true value. For most samples, this is unlikely.

As **maxplus5s** demonstrates a bias, we will eliminate it from consideration as the best rule of the four rules under investigation. At first glance, the remaining three rules seem to produce almost the same estimates, and it is hard to tell which ones might be better at estimating the total number of tanks. One way to help decide is to find the "error" in estimating the total number of tanks by subtracting the 245 tanks, or the parameter of the population, from an estimate.

10. Consider subtracting 245 from each of the estimates.
    a. What would a negative difference represent?

Sample Answers: A negative difference indicates the estimate was less than the actual value.

b. What does a value close to zero represent?

Sample Answers: Answers close to 0 indicate the estimate was close to the actual value.

c   How might you use the errors to figure out whether a rule is good or bad?

Sample Answers: Students might suggest adding all of the errors, taking the absolute value of the errors, or squaring the errors.

> **Teacher Tip:** You might ask students whether summing the errors and getting a sum of 0 would indicate a good estimator. They should recognize that the sum of the errors close to 0 does not indicate the rule is generating statistics close to the number of tanks. Several large positive values could be offset by several negative values. The cancellation around the value 0 indicates the need to address the sign of the errors.
> You might relate the concept of error in this context to the idea of residuals for those familiar with the term or introduce the term residual for a later study.

Move to page 1.9.

The spreadsheet on Page 1.9 displays the absolute values of
the errors for each rule.

11. If one rule were selected as a better rule for estimating the true
    value, how would the values displayed in the spreadsheet for
   this rule compare to the values of a rule that was not as good?

Sample Answers: The errors for rules that generally result in statistics close to the number of
tanks would have smaller absolute values than the other rules.

> **Teacher Tip:** The rules result in a range of estimates that have both large
> and small values. Students might not observe the subtle differences in the
> rules. Students will often, however, notice the generally smaller values with
> the rule **multimaxs**. Remind students that we eliminated **maxplus5s** due
> to the bias previously discussed. It is possible that **maxplus5s** could result
> in smaller errors than some of the remaining rules; however, **maxplus5a**
> consistently underestimates the true value. Our goal is to evaluate the
> remaining rules that do not demonstrate bias.

Move to page 1.10.

12. Examine the dotplots of the absolute value of the errors for
    each rule.
    a. How would you describe the variability displayed by each
        dotplot?
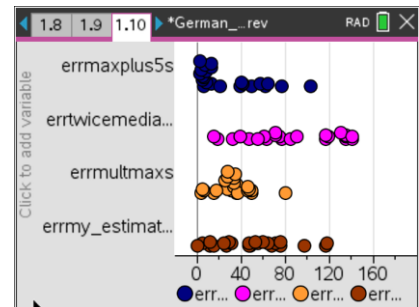
Sample Answers: The distributions except for **twicemedians** seem to be skewed right. The
dotplots of the absolute values generally indicate that **multimaxs** has the smallest range of
values, thus less variability. The rule **twicemedians** has the largest range of values of the rules,
thus a larger variability.

    b. Knowing that you only have one sample from which to estimate the total number of tanks,
        which rule—that does not appear biased—would you use? Explain your answer.

Sample Answers: Although answers could indicate different choices, generally students suggest
multimaxs as the better rule because of the smaller range and thus, less "error".

Move to page 1.11.

Page 1.11 provides a sum of the absolute value of the errors for
each of the rules.

13. Do the sums support your previous conjectures about the better
rules? Why or why not?

	A mean_error	B	C
=			
1	maxplus5	28.1304	
2	twicemedian	85.7826	
3	multmax	32.6696	
4	my_estimate	50.2696	
5			

A5

Sample Answers: Most of the simulations will indicate that the rule "multimaxs" results in the
smaller differences of the estimates from the population parameter. This results in smaller errors
and a smaller sum.

> **Teacher Tip:** The total error for the **maxplus5s** rule might be quite small
> but remind students that this rule was eliminated because it was a biased
> estimator. You might point students back to Page 1.8 for a visual reminder.

14. A rule that provides estimates with a small amount of variability around the actual number of tanks
would seem to be a good rule to select. Why is a small amount of variability important?

Sample Answers: If you have only one sample, a rule that you think would result in less
variability would provide an estimate that is less likely to be far from the true number of tanks. You
would have a better chance of being close to "right" with your estimate.

> **Teacher Tip:** Students usually are interested in further background
> information concerning the problem. The rule identified as **multmaxs**
> provides an estimate that is considered unbiased with relatively small
> variability around the actual value. This rule is similar to what was used in
> the actual work of the statisticians in World War II. The **multmaxs** used in
> this lesson is:
>
> $$N = [(n+1)/n]m$$
>
> This can also be rewritten as:
>
> $$N = m + (m/n)$$
>
> $N =$ estimated number of tank manufactured
>
> $n =$ number of tanks captured (in the examples for this lesson, n was 5)
>
> $m =$ largest serial number of the captured tanks.

The intelligence reports indicated that the number of tanks manufactured over a certain time period was actually over a thousand, while the statisticians indicated it was closer to 250. (See the table below.) It was verified after the war that the statistician's estimates were more accurate! Many military decisions were made based on an incorrect estimate of the number of tanks.

After the war, production records from the ministry of Albert Speer showed the actual number to be 255. Estimates of the number of tanks for some specific months for the statisticians, the spies in the field, and the German records are below.

Month	Statistical estimate	Intelligence estimate	German records
June 1940	169	1000	122
June 1941	244	1550	271
August 1942	327	1550	342

Source: http://en.wikipedia.org/wiki/German_tank_problem

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- Possible bias in a sample statistic that clearly under (or over) estimates the largest value.
- The variability in sampling distributions of sample statistics.
- The effectiveness of a rule in estimating the size of a population.

Assessment

1. Why would the statistic obtained from the rule **"=max(capture)"** be considered biased in estimating the number of German tanks?

   Sample Answers: The sample of captured serial numbers could include the largest serial number, although not likely. Thus, the estimate would be less than the true number of tanks except in the rare case of a sample that would actually capture the last tank manufactured for the month. As a result, this rule would consistently under estimate the number of tanks and demonstrates a bias similar to what was discussed in the activity.

2. Would you expect the sum of the absolute value of the errors to be large or small for the rule **"=2 x median(capture)"**? Explain

   Sample Answers: Compared to the other rules investigated, this rule resulted in a noticeably greater range of the sample statistics around the population parameter. It would result in larger errors (or residuals) and a relatively large sum of the absolute values.

3. Would there be an advantage for the problem faced in World War II in a rule that would generally over estimate or under estimate the number of tanks? Explain your answer.

   Sample Answers: Given the implications of the decision, a rule that would over estimate the number of tanks would better prepare the military. However, a rule that consistently produces a large overestimate would result in poor use of important resources.

TI-Nspire Navigator

Note 1
Question 6, Name of Feature: Screen Capture
Consider sharing students' dotplots using Screen Captures with TI-Nspire Navigator.

Note 2
Question 8, Name of Feature: Screen Capture
As each student's dotplots will be slightly different, this question provides an opportunity to use TI-Nspire Navigator to capture screen shots of your students' distributions. If using TINspire Navigator is not possible, have several students display their distributions either with a document camera or on an overhead projector and discuss the features of the distributions.