EXPLORATIONS

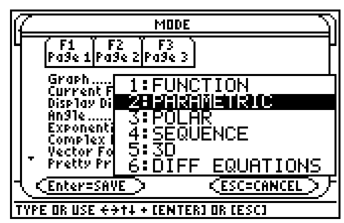
Chapter 8

Parametric, Vector, Polar, and 3D Functions

In this chapter, you will graph parametric functions, vectors, polar functions, and 3D functions on the TI-89.

Parametric functions

The motion of a particle moving in a plane can often be described by assuming the x- and y- coordinates are both functions of time. For example, $x = \cos(t)$, $y = \sin(t)$. Since x and y both have a common parameter t, these are called *parametric equations*. You can graph parametric functions on the TI-89 setting **Graph=PARAMETRIC** in the MODE dialog box. Also set **Angle=RADIAN**.

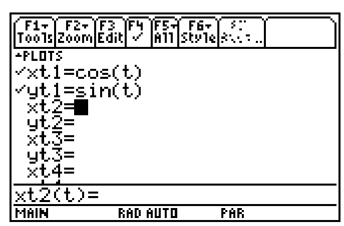


Example 1: Parametric equations for a circle

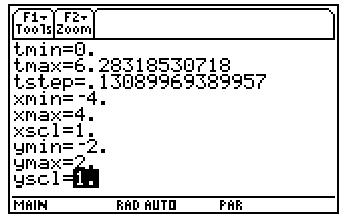
Graph the parametric equations $x = \cos(t), \quad y = \sin(t)$.

Solution

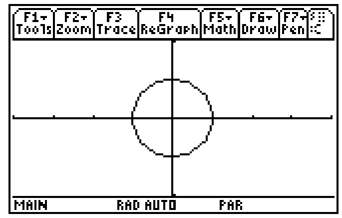
1. In the Y= Editor, clear all equations. Then enter the equations above in $xt1$ and $yt1$.



2. Press [◆] [WINDOW] to set the viewing window. The Window variable **tmin** (0) is the starting value for t and **tmax** (2π) is the final value. The increment in t from one point to the next in the graph is given by **tstep** ($\pi/24$). Enter the values for the Window variables as shown.



3. Press [◆] [GRAPH] to graph the parametric equations.



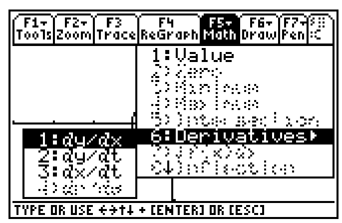
Example 2: Slope of a parametric curve, chain rule for parametric equations

Find the slope of the circle in Example 1 at $t = \frac{\pi}{4}$.

Solution

The slope of the curve is given by $\frac{dy}{dx}$. You can approximate this value from the MATH menu on the Graph screen.

1. With the graph from Example 1 displayed, press [F5] **Math** and select **6:Derivatives**.
2. Select **1:dy/dx**. The TI-89 returns to the graph and prompts for a t-coordinate.

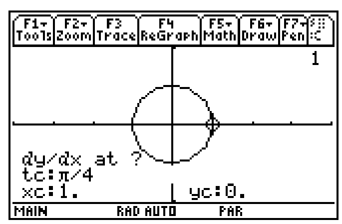


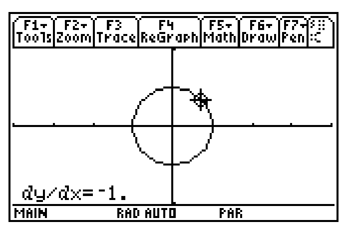
3. Enter the value.

   [2nd] [π] [÷] **4**

4. Press [ENTER].

   The slope of the curve is -1.



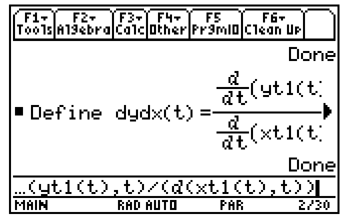


You can verify this result with the chain rule for parametric equations, which says that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

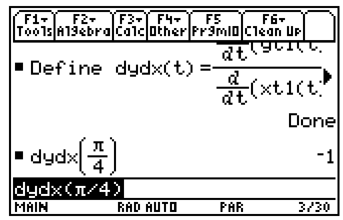
5. Return to the Home screen and define a function that gives the slope by entering:

   [CATALOG] **Define DYDX** [(] **T** [)] [=] [2nd] [d] **YT1** [(] **T** [)] [,] **T** [)] [÷] [2nd] [d] **XT1** [(] **T** [)] [,] **T** [)] [)]



6. Evaluate this function at $t = \frac{\pi}{4}$.

   The slope function you defined gives the same result as that returned by the GRAPH MATH *dy/dx* feature.



Example 3: Arc length of parametric curves

This example defines a function to calculate the arc length of a parametric curve.

Find the length of one arch of the cycloid $x=t-\sin(t)$, $y=1-\cos(t)$.

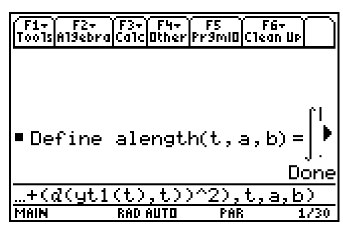
Solution

Arc length is given by the definite integral

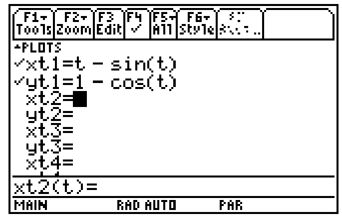
$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults.
2. Define the above integral with the command **Define alength .**

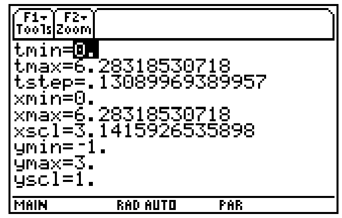
   [CATALOG] **Define ALENGTH** [(] **T** [,] **A** [,] **B** [)] [=] [2nd] [∫] [2nd] [√] [(] [2nd] [d] **XT1** [(] **T** [)] [,] [^] **2** [+] [(] [2nd] [d] **YT1** [(] **T** [)] [,] **T** [)] [)] [^] **2** [)] [,] **T** [,] **A** [,] **B** [)] [ENTER]



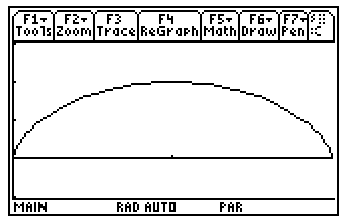
3. In the Y= Editor, enter the parametric equations for the cycloid.



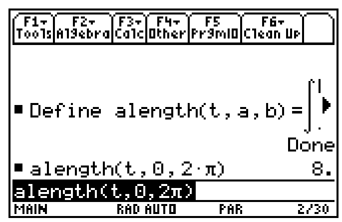
4. In the Window Editor, set the values as shown for a [0,2π] x [-1,3] window with **xscl** = π.



5. Press ◆ [GRAPH] to graph the arch.



6. Since one arch is completed for $0 \le t \le 2\pi$, return to the Home screen and enter the command **alength(t,0,2π)** to find the arc length of the arch.



The length of the arch is 8.

Example 4: Parametric equations for trajectories

In this example, you will revisit the baseball problem from Chapter 7, Example 5. This time the goal is to find the parametric equations for the position of the baseball from the information given about acceleration.

A baseball is hit when it is 3 feet above the ground. It leaves the bat with an initial velocity of 152 ft/sec at an angle of 20° with the horizontal. How far will the ball travel?

Solution

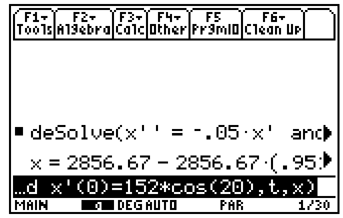
The baseball moves both horizontally and vertically. Assume that the horizontal deceleration due to air resistance is directly proportional to the horizontal velocity. The vertical acceleration due to gravity is -32 ft/s^2, and the vertical deceleration due to air resistance is proportional to the vertical velocity. Assume a constant of proportionality for air resistance of -0.05.

You can find the parametric equations for the position of the baseball by solving the differential equations

$x'' = -.05x'$, $x(0)=0$, $x'(0)=152\cos(20°)$

$y'' = -32 - .05y'$, $y(0)=3$, $y'(0)=152\sin(20°)$

1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults. In the MODE dialog box, set **ANGLE=DEGREE**.

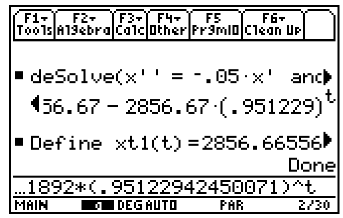


2. Solve the first differential equation with the command **deSolve(x″= -.05x′ and x(0)=0 and x′(0)=152cos(20),t,x)**.

   [CATALOG] **deSolve(X** [2nd] [′] [2nd] [′] [=] [(-)] **.05X** [2nd] [′] [CATALOG] **and X** [(] **0** [)] [=] **0** [CATALOG] **and X** [2nd] [′] [(] **0** [)] [=] **152** [×] [2nd] [COS] [(] **20** [)] [,] **T** [,] **X** [ENTER]

   The solution is x=2856.67–2856.67(.951229)^t.

3. Store this in $xt1$ with the **Define** command.

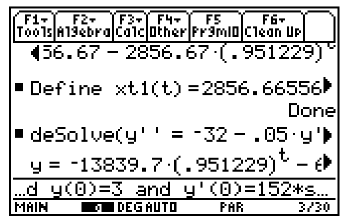


   [CATALOG] **Define XT1**[(] **T** [)] [=] [↑] [ENTER]

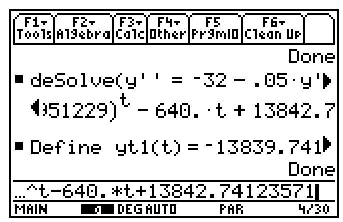
   Press [←] to move left and then press [⇐] to delete **x=**.

4. Solve the second differential equation with the command **deSolve(y″= -32 -.05y′ and y(0)=3 and y′(0)=152sin(20),t,y)**.

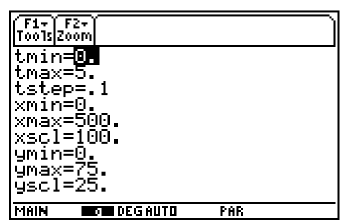
   The solution is y=–13839.7(.951229)^t–640t+13842.7



5. Store this solution in $yt1$ with the command **Define yt1(t)=-13839.7(.951229)^t-640t+13842.7**. Paste the result of the last command to simplify entering this command.

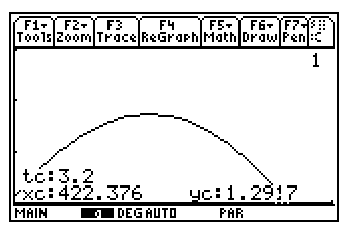


6. In the Window Editor, set up the viewing window as shown.



7. Graph the parametric equations and trace to see how far the baseball traveled.

   The ball travels about 422 feet.



Vectors

If $\boldsymbol{r} = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$ is the position vector for a projectile, the velocity vector is given by $\boldsymbol{v} = x'(t)\boldsymbol{i} + y'(t)\boldsymbol{j}$ and the acceleration vector is $\boldsymbol{a} = x''(t)\boldsymbol{i} + y''(t)\boldsymbol{j}$. In this example, one-dimensional matrices are used to represent vectors on the TI-89.

Example 5: Symbolic and graphical representation of vectors

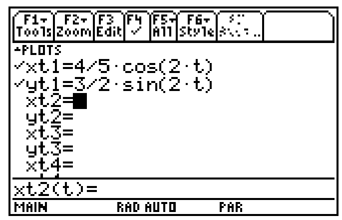
Represent the position vector

$$\boldsymbol{r}(\mathrm{t}) = \frac{4}{5}\cos(2\mathrm{t})\boldsymbol{i} + \frac{3}{2}\sin(2\mathrm{t})\boldsymbol{j}$$

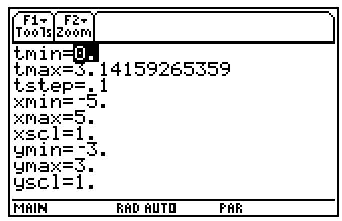
and the velocity and acceleration vectors symbolically and graphically.

Solution

1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults. In the MODE dialog box, set **Angle=RADIAN**.
2. In the Y = Editor, enter the parametric equations that correspond to the position vector as $xt1$ and $yt1$.



3. In the Window Editor, set up the viewing window as shown.

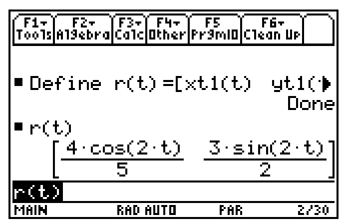


4. Return to the Home screen and define the *position vector* function **r**(t) with the command **Define r(t)=[xt1(t),yt1(t)]**.

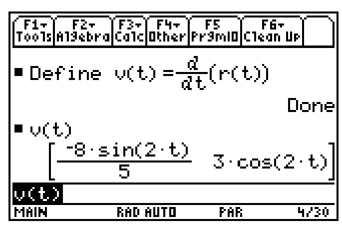
   [CATALOG] **Define R** [(] **T** [)] [=] [2nd] [[] **XT1** [(] **T** [)] [,] [(] **YT1** [(] **T** [)] [2nd] []] [ENTER]

5. Display $\boldsymbol{r}(t)$.

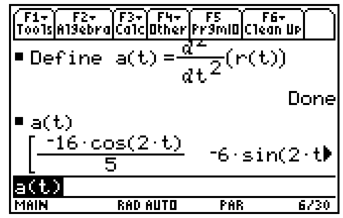
   R [(] **T** [)] [ENTER]



6. Define the *velocity vector* as the first derivative of the position vector with the command **Define v(t)=*d*(r(t),t)** and display $\boldsymbol{v}(t)$.

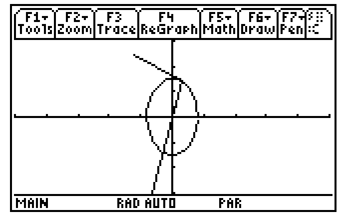


7. Define the *acceleration vector* as the second derivative of the position vector with the command **Define a(t)=** *d***(r(t),t,2)**, and display $\boldsymbol{a}(t)$.



8. Now enter the following commands to graph each of the vectors for t = .6. You can enter the **Line** command one letter at a time or copy it from the CATALOG.

```
.6→c
r(c)[1,1]→p
r(c)[1,2]→q
Line 0,0,p,q
Line p,q,p+v(c)[1,1],q+v(c)[1,2]
Line p,q,p+a(c)[1,1],q+a(c)[1,2]
```



Polar functions

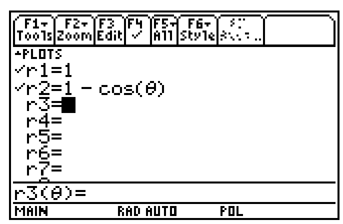
You must change the graphing mode to **POLAR** to graph polar functions.

Example 6: Area bounded by polar curves

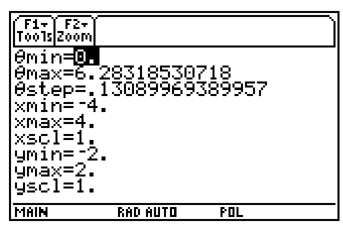
Use polar graphing to find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Solution

1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults. In the MODE dialog box, set **GRAPH=POLAR**.
2. Press [◆] [Y=] to display the Y= Editor. Then enter the equations in $r1$ and $r2$. Press [◆] [^] to enter **θ**.

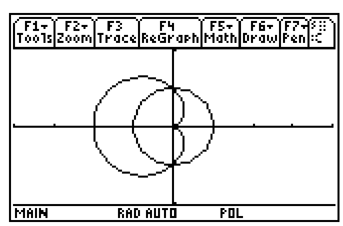


3. In the Window Editor, set up the viewing window as shown.



4. Graph the polar curves.

   From the graph, it appears that the polar functions intersect at $\theta = -\frac{\pi}{2}$ and $\frac{\pi}{2}$.

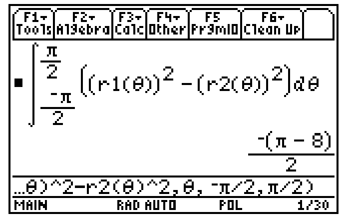


5. The area is given by entering the definite integral ∫ **(r1(θ)^2- r2(θ)^2,θ,-π/2, π/2)** on the Home screen and then dividing the result by 2.

   [2nd] [∫] **R1** [(] [◆] [θ] [)] [^] **2** [−] **R2** [(] [◆] [θ] [)] [^] **2** [,] [◆] [θ] [)] [,] [(-)] [2nd] [π] [÷] **2** [,] [2nd] [π] [÷] **2** [)] [ENTER]

   The area is

$$\frac{8-\pi}{4}$$



3D functions

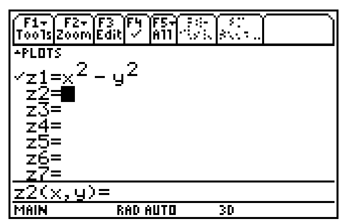
To graph functions of two variables, select 3D in the MODE dialog box.

Example 7: The graph of a saddle

Using 3D graphing, graph the "saddle" $z = x^2 - y^2$ in a [-2,2] x [-2,2] x [-2,2] viewing window.

Solution

1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults. In the MODE dialog box, set **GRAPH=3D**.

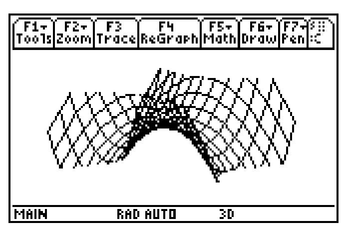


2. Press [◆] [Y=] to display the Y= Editor. Enter the equation in $z1$.

3. Press [◆] [WINDOW] and enter the following window values:

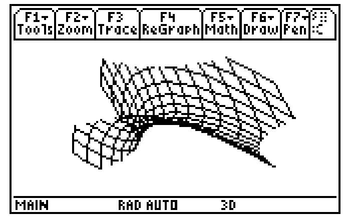
eyeθ= 30	**ymin= -2**
eyeϕ= 70	**ymax= 2**
eyeψ= 0	**ygrid= 14**
xmin= -2	**zmin= -2**
xmax= 2	**zmax= 2**
xgrid= 14	**ncontour= 5**

4. Press [◆] [GRAPH] to see the surface.



5. You can rotate the surface with the cursor movement keys.

6. Return to the Window Editor to see how the rotations affect the viewing angle variables (**eyeθ, eyeϕ,** and **eyeψ,** for example).



Exercises

1. Given the parametric equations $x = 3\cos(t)$ and $y = 2\sin(t)$ for $0 \le t \le 2\pi$:

   (a) Graph the curve described by these parametric equations.
   (Use viewing window [-3,3] x [-6,6]).

   (b) Use the **Math** menu on the Graph screen to find the slope of the parametric curve at

   $t = \dfrac{\pi}{4}$.

   (c) Use the chain rule for parametric equations to find the slope of the parametric curve at $t = \dfrac{\pi}{4}$.

   (d) Find the length of the parametric curve.

2. Given the parametric equations $x = \sin(2t)$ and $y = \sin(t)$ for $0 \le t \le 2\pi$.

   (a) Graph the curve described by these parametric equations.
   (Use viewing window [-4,4] x [-2,2]).

   (b) Use the **Math** menu on the Graph screen to find the slope of the parametric curve at $t = \dfrac{\pi}{2}$.

   (c) Use the chain rule for parametric equations to find the slope of the parametric curve at. $t = \dfrac{\pi}{2}$.

   (d) Find the length of the parametric curve.

3. A baseball is hit when it is 3 feet above the ground. It leaves the bat with an initial velocity of 127 ft/sec at an angle of 30° with the horizontal. The baseball moves both horizontally and vertically. Assume that the horizontal deceleration due to air resistance is directly proportional to the horizontal velocity. The vertical acceleration due to gravity is 32 ft/s^2 and the vertical deceleration due to air resistance is proportional to the vertical velocity. Assume a constant of proportionality for air resistance of -0.05.

   Find the parametric equations for the motion of the ball, and use the parametric graph to determine how far will the ball travel.

4. Represent the position vector $r = 4\cos(t)\, i + 3\sin(t)\, j$ and the velocity and acceleration vectors at ***t***=0.6 symbolically and graphically. (Use viewing window [-10,10] x [-5,5].)

5. Use polar graphing to find the area inside the circle $r = 2\sin(\theta)$ and outside the cardioid $r=1-\cos(\theta)$.

6. Graph the following three-dimensional surfaces.

   (a) $z = 9 - x^2 - y^2$

   (b) $z = x * \sin(y)$

   (c) $z = \cos(x * y)$