Geometric Series

ID: 10224

Time required
40 minutes

Activity Overview

Students will use TI-Nspire technology to explore infinite geometric series and the partial sums of geometric series. The students will also determine the limits of these sequences and series using tables and graphs.

Topic: Series

- *Derive and apply a formula for the sum of an infinite convergent geometric series.*
- *Use the $\sum$ template to verify the formula for the sum of an infinite series in specific cases.*
- *Prove and apply the ratio (of consecutive terms) test to prove a series convergent or divergent.*
- *Prove that a necessary condition that a geometric series converges is that $|r| < 1$ where r is the common ratio.*

Teacher Preparation and Notes

- *Students should already be familiar with sequences, partial sums, and the definition of convergent and divergent series.*
- *This activity is intended to be a* ***student-center*** *exploration. Problems 3 and 4 promote a class discussion.*
- *The student is intended to guide students through the main ideas of the activity. It also serves as a place for students to record their answers. Alternatively, you may wish to have the class record their answers in the student .tns file, on separate sheets of paper, or just use the questions posed to engage a class discussion. Students should be familiar with the capabilities and tools of the handheld being used in this activity.*
- *Notes for using the TI-Nspire™ Navigator™ System are included throughout the activity. The use of the Navigator System is not necessary for completion of this activity. Questions are in "Exam" mode. If you are not using a Navigator to automatically grade the activity, you may want to change them "Self-Check" so students can get immediate feedback. With the Teacher Software, on a Question page, like page 1.2, choose Teacher Tool Palette under the menu options. Then click "Question Properties …" and change the Document Type in the pop-up dialogue box.*
- ***To download the student TI-Nspire document (.tns file) and student worksheet, go to education.ti.com/exchange and enter "10224" in the keyword search box.***

Associated Materials

- *GeometricSeries_Student.doc*
- *GeometricSeries.tns*

Infinite series

An infinite series can be defined as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$, where a_1, a_2, and a_3 are terms of the series. It is beneficial to students if they are already introduced to sequences.

Have students complete the exercises on page 1.3–1.5. They can also use the .tns file or the student worksheet to record their answers.

1. Find the next three terms of the infinite series.

a. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ **b.** $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots$ **c.** $2 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32} \ldots$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$ $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7}$ $2 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32} + \frac{81}{128} + \frac{243}{512}$

Students should see for question **1.c.** that the first term 2 is the same as *a* in a geometric series, meaning $a + ar + ar^2 + \ldots + ar^n + \ldots$

2. Write an expression in terms of *n* that describe each of the above series using sigma notation.

a. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ or $\sum_{n=1}^{\infty} \frac{1}{2^n}$ **b.** $\sum_{n=1}^{\infty} \frac{n}{n+1}$ **c.** $\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^{n-1}$

TI-Nspire Navigator Opportunity: *Quick Poll, Live Presenter*

See Note 1 at the end of this lesson.

Finding the sum of a geometric series

Using the *Calculator* application, students can find the partial sum of a geometric series. In this problem, students will find the sixth partial sum of two geometric series. To find the sum of the series go to the calculator page, press [template key] and select the sum template or select the **Sum** command from the **Calculus** menu. The bottom should read *n* = 1, the top number should be 6, and the function should be entered in the parentheses.

3. $\sum_{n=1}^{6} \left(\frac{1}{2}\right)^n = \frac{63}{64} = 0.984375$ **4.** $\sum_{n=1}^{6} 2\left(\frac{3}{4}\right)^{n-1} = \frac{3367}{512} = 6.57617$

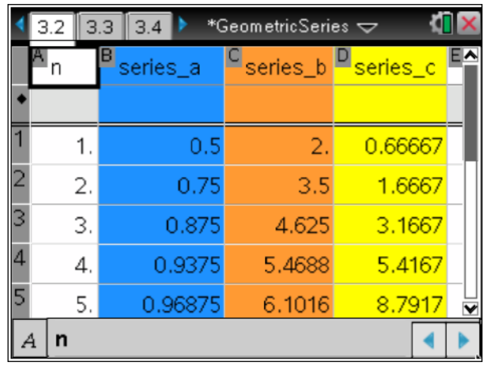
Convergence and divergence of geometric series

A geometric series with first term *a* and common ratio *r* is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots + ar^n + \ldots, a \neq 0.$$

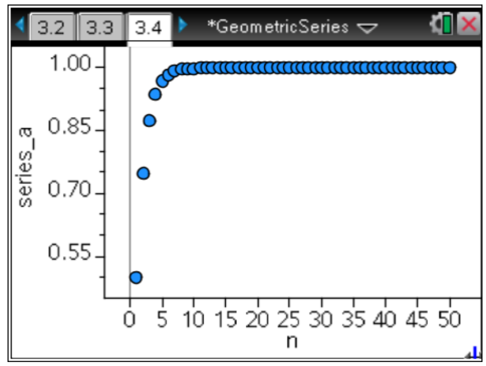
A geometric series diverges if $|r| \geq 1$. It converges to the sum $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ if $0 < |r| < 1$.

These conditions must be used in determining whether a series diverges or converges. It is also worth noting that a series may diverge, but will not necessarily diverge to infinity. A value *r* that is less than –1 will result in a series that diverges and has terms whose signs alternate from positive to negative, not diverging to infinity.



n	series_a	series_b	series_c
1.	0.5	2.	0.66667
2.	0.75	3.5	1.6667
3.	0.875	4.625	3.1667
4.	0.9375	5.4688	5.4167
5.	0.96875	6.1016	8.7917

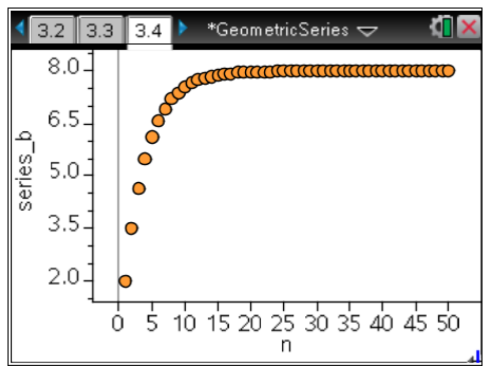
Another important note to students is that a series converges or diverges if the sequence of the partial sums converges to its sum or diverges.



To find the sum of a geometric series, students must take the limit of the *n*th sum. The series $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$, is a special case of the geometric series. One way of attempting this problem is to list the partial sums of the series then determine the *n*th sum.

The partial sums for this series would be

$S_1 = \frac{1}{2},\ S_2 = \frac{3}{4},\ S_3 = \frac{7}{8},\ S_4 = \frac{15}{16},\ \ldots$

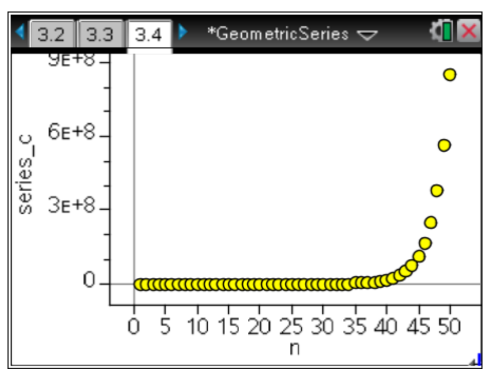


Guide students so they see that the denominator is 2 raised to the *n*th power and the numerator is always 1 less than the denominator. This gives $\frac{2^n - 1}{2^n}$.

Take the limit of the sum:

$$\lim_{n\to\infty}\frac{2^n - 1}{2^n} = 1$$

By the geometric series test, the series converges because $0 < \frac{1}{2} < 1$, and the sum is 1, by finding the limit.



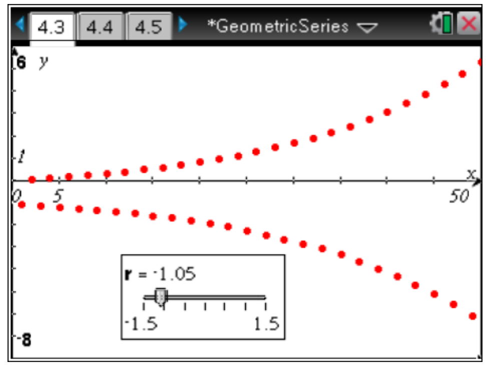
Another way is to use the geometric series test that can be stated $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ or $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. It is important to note to students the index of the series.

The graphs and lists of the first 50 terms of three series are shown above. The graphs show what these values are approaching.

TI-Nspire Navigator Opportunity: *Screen Capture, Collect, Review, Save to portfolio*
See Note 2 at the end of this lesson.

Extension

Instruct students to move to page 4.3 and drag the point that changes the *r*-value and the graph. Ask students to observe the series converge when $|r| < 1$ and diverges elsewhere.



TI-Nspire Navigator Opportunities

Note 1
Problem 1, *Quick Poll, Live Presenter*

Quick Polls can be used on any page that has a question, like 1.5, 1.6, 1.8, 2.1, 2.2, 2.3, and 4.4, 5, 6.
Live Presenter can be used to have a student demonstrate the use of the minimized slider. A student can simply click on the arrows next to n and then use the arrows on the touchpad to change it.

Note 2
Problem 2, *Screen Capture, Collect, Review, Save to Portfolio*

Screen Capture can be used to check for understanding. After checking that students can change the graph on page 3.4 to see the other series, ask the class to show you the graph of a converging series or 'Show a diverging series." Ctrl 0 on your computer keyboard will fit the class screen capture on the available screen space.

After the activity is completed, collect the file and recap what was learned by reviewing students' answers in the review workspace. This formative assessment can be documented by saving the file to the portfolio where the grade will automatically be calculated.