Investigating Properties of the Diagonals of Quadrilaterals

Definitions:

- **Quadrilateral—**a four-sided polygon.
- **Diagonal of a quadrilateral**—a segment that connects opposite vertices.
- **Kite**—a quadrilateral with two distinct pairs of consecutive sides congruent.
- **Trapezoid—**a quadrilateral with at least one pair of parallel sides.
- **Isosceles trapezoid**—a trapezoid with one pair of base angles congruent.
- **Parallelogram**—a quadrilateral with two pairs of sides parallel.
- **Rectangle**—a quadrilateral with four right angles.
- **Rhombus**—a quadrilateral with four congruent sides.
- **Square—**a quadrilateral with four congruent sides and four right angles.

Construct and Investigate:

1. Figure 1 shows a quadrilateral hierarchy in which each quadrilateral shares all the properties of those connected above it in the chart. Construct each figure in the chart, and verify that the definitions allow each type of quadrilateral to be a special case of the quadrilaterals connected above it in the chart.
2. For each figure, write as many conjectures as possible about its diagonals. Be sure that the statements are true for the figure and all connected below it but not true for the figures connected above it in the chart.

   For example, the diagonals of an isosceles trapezoid are congruent. This is also true for rectangles and squares but not for trapezoids or quadrilaterals.

   Each property should appear only once in your list, associated with the figure that first displays the property in the hierarchy chart.

   Use the Voyage™ 200 with Cabri to assist in the investigations. Examine relationships involving lengths, angle measures, symmetry, bisection, collinearity, areas, and whatever else comes to mind.

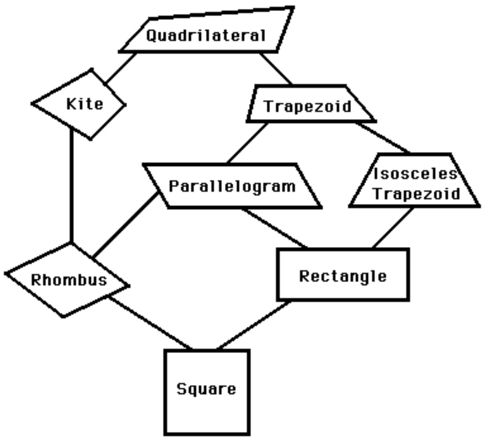

Figure 1

Explore:

1. Investigate the relationship between the four triangles formed by the diagonals of any convex quadrilateral. See whether you can discover a relationship that is always true by doing some creative explorations and measurements.
2. Carpenters and other builders use a property of the diagonals of a rectangle to make sure that they have four right angles when they build a wall or square up the foundation of a house. See whether you can discover which property they use. Check with someone in the building trades in your community to verify that this property is actually used. Ask your contact person to share other ways in which geometry is used in construction.

Teacher’s Guide: Investigating Properties of the Diagonals of Quadrilaterals

Construct and Investigate:

The definitions and hierarchy chart in this activity are from *Geometry* (Coxford, Usiskin, and Hirschhorn, 1991). These definitions may not be entirely consistent with other textbooks. In particular, quadrilaterals and kites are not necessarily convex, and trapezoids are defined as having at least one pair of parallel sides. In this activity quadrilaterals are defined as polygons, and therefore, crossed figures with four sides are not considered quadrilaterals. You may want students to use a different set of definitions than those presented here. You should be aware, however, that such definitions will change some of the statements and/or their location in the hierarchy listed below.

This activity takes considerable time to complete if each student or group of students does the entire exploration. An alternative technique may be to assign different groups different quadrilaterals to construct and investigate. The groups could then report their findings and assemble a composite list. Resolve conflicts with either large group discussion, or small student groups. You can assign certain quadrilaterals to particular groups or use a random selection process. The properties of some quadrilaterals appear more difficult to find than others.

There are many ways to construct a particular quadrilateral besides using the definition. It may be worthwhile to have students determine different constructions based on properties of various quadrilaterals. For more information about kites and trapezoids, see the Teacher’s Guides for the activities *Investigating Properties of Kites* and *Investigating Properties of Trapezoids*.

The properties of diagonals listed for each quadrilateral are valid for that figure and for all quadrilaterals connected below it in the hierarchy chart (Figure 1). These properties include:

Quadrilaterals

- The quadrilateral has two diagonals.
- The sum of the lengths of the diagonals is less than the perimeter of the quadrilateral.
- The sum of the lengths of the two diagonals is equal to the perimeter of the quadrilateral formed by connecting the midpoints of the sides of the original quadrilateral.

Kites

- The diagonals of the quadrilateral are perpendicular.
- At least one diagonal of the quadrilateral is the bisector of the other.
- At least one diagonal bisects two angles of the quadrilateral.
- At least one diagonal lies on a symmetry line of the quadrilateral.
- One-half the product of the diagonals is equal to the area of the quadrilateral.

Trapezoids

- The diagonals of the quadrilateral intersect. (This property is true for all convex quadrilaterals, but this is the first level in the hierarchy where you are assured that the quadrilateral is convex.)
- The products of the areas are equal for opposite pairs of triangles formed by the intersection of the diagonals of a quadrilateral. (This property is true for all convex quadrilaterals, but this is the first level in the hierarchy at which you are assured that the quadrilateral is convex.)
- The angles formed between a diagonal of the quadrilateral and its bases are congruent.
- The diagonals divide the quadrilateral into four triangles such that the areas are equal for at least one pair of these triangles (Figure 2).
- The intersection point of the diagonals is collinear with the midpoints of the bases of the quadrilateral (Figure 3).
- For both diagonals of the quadrilateral, the ratio of the distances from the midpoints of the bases to the intersection point of the diagonals is the same as the ratio of the distances from the vertices to the diagonal intersection point.

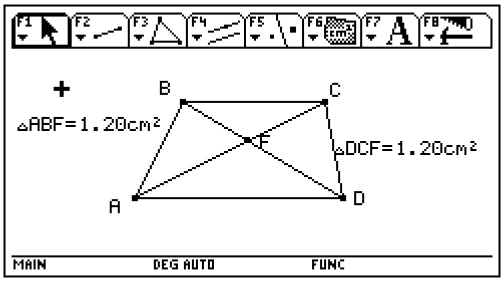

Figure 2

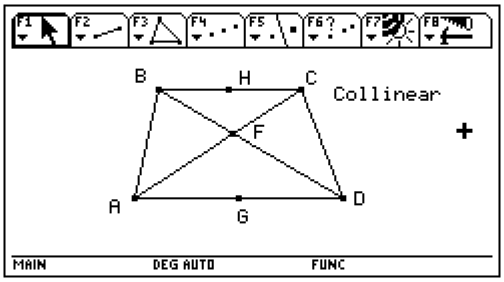

Figure 3

Isosceles Trapezoids

- The diagonals of the quadrilateral are congruent.
- The diagonals divide the quadrilateral into three pairs of congruent triangles and one pair of similar isosceles triangles.
- At least one symmetry line of the quadrilateral contains the intersection point of the diagonals.

Parallelograms

- Each diagonal divides the quadrilateral into two congruent triangles.
- The diagonals of the quadrilateral bisect each other.
- The diagonals divide the quadrilateral into two pairs of congruent triangles.
- The intersection point of the diagonals is also the intersection point of the lines connecting the midpoints of the opposite sides of the quadrilateral (Figure 4).
- The diagonals meet at the centroid of the quadrilateral (Figure 4).
- The intersection point of the diagonals is a bisector of the segment connecting the intersection points of the perpendicular bisectors of the sides of the quadrilateral (Figure 4).
- The diagonals divide the quadrilateral into four triangles of equal area (Figure 5).

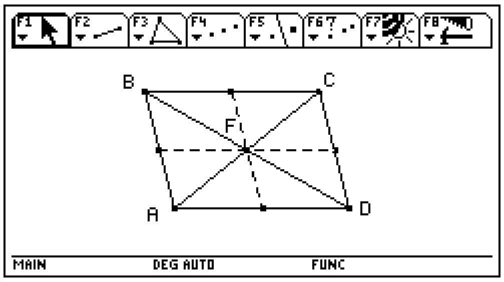

Figure 4

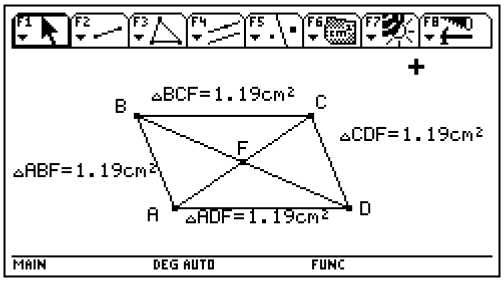

Figure 5

Rectangles

- The diagonals of the quadrilateral are congruent and bisect each other.
- The intersection point of the diagonals is also the intersection point of the perpendicular bisectors of the sides of the quadrilateral.
- The length of each diagonal is equal to the square root of the sum of the squares of two consecutive sides of the quadrilateral.

Rhombi

- Both diagonals bisect opposite angles of the quadrilateral.
- Both diagonals form symmetry lines for the quadrilateral.
- The diagonals divide the quadrilateral into four congruent right triangles.
- The intersection point of the diagonals is the incenter of the quadrilateral.
- The diagonals lie on the symmetry lines of the quadrilateral.

Squares

- The diagonals divide the quadrilateral into four congruent isosceles right triangles.
- The intersection point of the diagonals is the circumcenter of the quadrilateral.
- The diagonals meet at the symmetry point of the quadrilateral.
- Any line through the intersection of the diagonals divides the perimeter and the area of the quadrilateral into two equal parts.
- The ratio of the length of a diagonal to a side is equal to $\sqrt{2}$.

Explore:

1. The products of the areas of opposite triangles formed by the diagonals of a convex quadrilateral are always equal (Figure 6).
2. When laying out the framing for a wall or the foundation of a rectangular home, builders measure the diagonals of the parallelogram and adjust the structure until the lengths of the diagonals are congruent. This assures that the four corners are square and the wall or foundation is rectangular (Figure 7).

   Another technique for squaring a wall uses the properties of a right triangle with legs of 3 and 4 units and a hypotenuse of 5 units (or multiples of these values).

   A carpenter's square, or framing square, is a tool used to cut rafters, steps, and braces as well as make right-angle cuts. This tool uses the properties of right triangles using trigonometry and the **Pythagorean Theorem**.

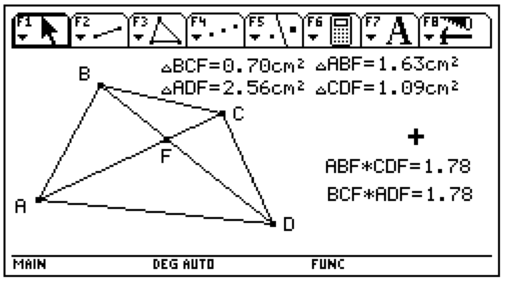

Figure 6

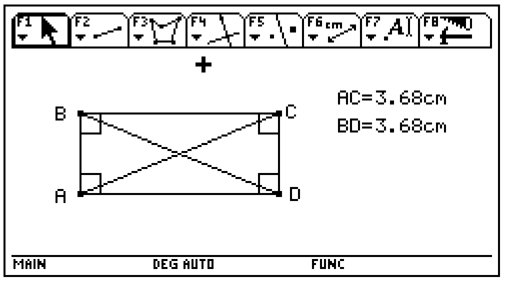

Figure 7

