

## Chapter 3

### Higher Order Differential Equations

The differential equations you solved in Chapter 1 and Chapter 2 are called *first-order differential equations* because they involved first derivatives. An equation that contains a second derivative is called a *second-order differential equation*.

#### Introduction

Although the TI-86 only allows entry of first order differential equations, you can graph the solution to a second-order differential equation by making substitutions that will reduce it to a system of first order differential equations.

These substitutions are:

Let  $y = Q1$ . Then  $y' = Q'1$  and  $y'' = Q''1$ . Since there is no  $Q''1$  in the differential equation editor ( $Q'(t)=$  screen), let  $Q'1 = Q2$  and then  $y'' = Q''1 = Q'2$ . Summarize these substitutions as follows:

$$y = Q1$$

$$y' = Q'1 = Q2$$

$$y'' = Q''1 = Q'2$$

This means that once you set up  $Q'1 = Q2$  on the TI-86, you can enter the expression for the second derivative in  $Q'2$ .

#### Example 1: Acceleration Due to Gravity

A ball is dropped from an initial height of 2 meters. The acceleration of the ball is  $-9.8 \text{ m/s}^2$ . How long will it take to hit the ground? How fast will it be going when it hits the ground?

**Solution**

Let  $y$  equal the height of the ball. Then the velocity is

$$\frac{dy}{dt}$$

and the acceleration is

$$\frac{d^2y}{dt^2} = -9.8$$

You can reduce the second-order differential equation for acceleration to a system of first-order equations with substitutions similar to those described in the introduction.

$$\begin{aligned} \text{height} &= y = Q1 \\ \text{velocity} &= y' = Q'1 = Q2 \\ \text{acceleration} &= y'' = Q''1 = Q'2 = -9.8. \end{aligned}$$

1. Enter the differential equation for acceleration in the TI-86 by letting  $Q'1 = Q2$  and  $Q'2 = -9.8$ . Since you have equations for  $Q'1$  and  $Q'2$ , you will see graphs of  $Q1$  and  $Q2$ .
2. Set the graphing style to thick for  $Q'1$  and thin for  $Q'2$  so you can tell which graph is height and which graph is velocity.

Change the graphing styles with the **STYLE** feature. (Figure 3.1)



Figure 3.1

3. Turn the slope field off using the format screen (**MORE**) **[F1]** (**FORMAT**). (Figure 3.2)

If you don't turn the field off, you will get an error message.



Figure 3.2

4. Select the axes editor and let  $x = t$ . In order to see both height ( $Q1$ ) and velocity ( $Q2$ ) you must let  $y = Q$ . If you were to let  $y = Q1$  in the axes editor, you would only see the graph of  $Q1$ . (Figure 3.3)



Figure 3.3

5. Select an appropriate viewing window. You expect the ball to drop for about 1 second, so choose  $[0,1]$  for both **tMin/tMax** and **xMin/xMax**. Choosing **.01** for **tStep** lets you see the ball every hundredth of a second. You need values for **yMin/yMax** that will show both height and velocity. Since the height will change from 2 to 0 and the velocity will change from 0 to about  $-10$  in the first second, choose the interval  $[-10, 2]$  for **yMin/yMax**. (Figures 3.4 and 3.5)

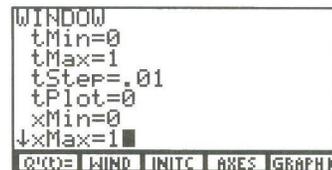


Figure 3.4



Figure 3.5

6. Enter the initial conditions with the initial conditions editor. (The value for **tMin** was entered in the window editor.) Since the initial height is 2, let **Q11** = 2. Since the ball started from rest, let the initial velocity = **Q12** = 0. (Figure 3.6)



Figure 3.6

7. Press **F5** (**GRAPH**) to see the graph. (Figure 3.7)

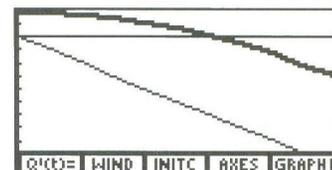


Figure 3.7

The thick graph is **Q1** (height) and the thin graph is **Q2** (velocity). You can use the **TRACE** feature to see when the  $y$ -coordinate of **Q1** is close to zero. This will tell you when the ball hits the ground. The closest you can get to a  $y$ -coordinate of 0 in this viewing window is  $-.00704$ . (Figure 3.8)

From the graph, you can estimate that the ball hit the ground after about 0.64 seconds.

Now press **▾** to move the trace cursor to the corresponding graph of **Q2**. Notice that the number in the upper right corner of the screen changed from 1 to 2. This is because you moved the trace cursor from the graph of **Q1** to the graph of **Q2**. (Figure 3.9)

The ball's velocity was about  $-6.272$  m/sec when it hit the ground.

In the next example, you will use this same method of substitution to graph the solution to a *third-order differential equation*.

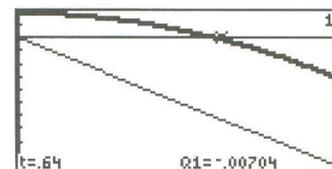


Figure 3.8

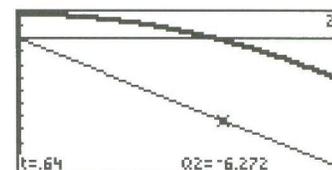


Figure 3.9

## Example 2: A Third-Order Differential Equation

Graph the solution to the differential equation:

$$y''' - y' = 4e^{-x} + 3e^{2x}$$

with initial conditions  $x = 0$ ,  $y = 0$ ,  $y' = -1$ , and  $y'' = 2$ .

### Solution

Reduce this third-order differential equation to a system of first order equations with the following substitutions:

$$y = Q1$$

$$y' = Q'1 = Q2$$

$$y'' = Q''1 = Q'2 = Q3$$

$$y''' = Q'''1 = Q''2 = Q'3$$

1. If you let  $Q'1 = Q2$  and  $Q'2 = Q3$ , you can enter  $y'''$  in  $Q'3$ . You solve the differential equation for  $y'''$ , obtaining

$$y''' = 4e^{-x} + 3e^{2x} + y'.$$

Enter the expression for  $y'''$  in  $Q'3$  using  $t$  for  $x$  and  $Q2$  for  $y'$ .

2. Since you only want to see the graph of  $Q1$ , you need to deselect  $Q'2$  and  $Q'3$ . You can select or deselect an equation for graphing by moving the cursor to the appropriate line in the differential equation editor ( $Q'(t) =$  screen), and pressing  $[F5]$  (**SELECT**). When an expression is deselected, the equal sign is no longer highlighted. Select a thin graphing style for  $Q'1$ . (Figure 3.10)

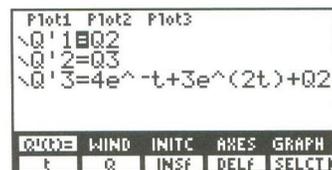


Figure 3.10

3. Since  $y = Q1$ ,  $y' = Q2$ , and  $y'' = Q3$ , enter the initial conditions  $x = 0$ ,  $y = 0$ ,  $y' = -1$ , and  $y'' = 2$  as  $tmin = 0$ ,  $Q11 = 0$ ,  $Q12 = -1$  and  $Q13 = 2$ . (Figure 3.11)



Figure 3.11

4. Enter these values for the viewing window:  $tMin = 0$ ,  $tMax = 2$ ,  $tStep = 0.1$ ,  $tPlot = 0$ ,  $xMin = -2$ ,  $xMax = 2$ ,  $xScl = 1$ ,  $yMin = -5$ ,  $yMax = 10$ ,  $yScl = 1$ ,  $difTol = .001$ .

The graph of  $y = f(x)$  is shown in Figure 3.12.

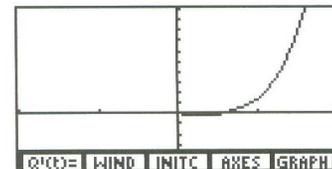


Figure 3.12

5. You can compare the solution you just found with the exact solution

$$y = -\frac{9}{2} + 4e^{-x} + 2xe^{-x} + \frac{1}{2}e^{2x}$$

by using the **DrawF** command in the GRAPH DRAW menu. (Figures 3.13 and 3.14)

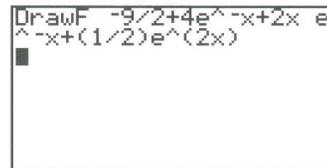


Figure 3.13

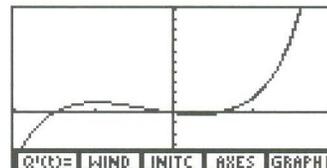


Figure 3.14

Notice that the graph of **Q1** only appears to the right of the initial value of **x** while the exact solution was drawn from **xMin** to **xMax**. Both graphs appear to coincide to the right of the initial point, although there are slight differences because the differential equation solver uses an approximate numerical method.

### Example 3: LRC Circuits

An electric circuit with a resistor, capacitor and inductor in series with an electromotive force is called an LRC circuit where  $R$  is the resistance in ohms,  $C$  is the capacitance in farads, and  $L$  is the inductance in henrys. The electromotive force is  $E$  (volts). When the circuit is completed, the capacitor will charge up. Kirchhoff's Law says that the charge  $Q$  on the capacitor (measured in coulombs) satisfies the second order differential equation:

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = E .$$

The rate at which the charge on the capacitor changes is equal to the current in the circuit. This current (measured in amperes) is represented by  $I$  so

$$\frac{dQ}{dt} = I .$$

If  $E = 1$ ,  $R = 2$ ,  $L = 1$ , and  $C = \frac{1}{2}$ , make a table of values that show how  $Q$  and  $I$  change over time.

**Solution**

You need to find a numeric solution for the second order differential equation

$$\frac{d^2Q}{dt^2} + 2\frac{dQ}{dt} + 2Q = 1.$$

Let

$$Q = Q1,$$

$$\frac{dQ}{dt} = Q'1 = Q2,$$

$$\frac{d^2Q}{dt^2} = Q''1 = Q'2.$$

When you solve the second-order differential equation for

$$\frac{d^2Q}{dt^2},$$

you obtain

$$\frac{d^2Q}{dt^2} = 1 - 2Q - 2\frac{dQ}{dt}.$$

1. You can enter this second order differential equation as a system of first order equations. Since you just entered a third order system in the previous example, you may have an expression stored in **Q'3**. Move the cursor to this expression and press **[F4] (DELf)** to remove it. (Figure 3.15)
2. Since **Q1** is charge and **Q2** is current, their initial values will be zero. The initial conditions editor may show a value for **Q13** left over from a previous problem. This value doesn't affect the graph. Only the initial variables with a square beside them are active for this graph. (Figure 3.16)
3. Since you want to look at **Q** and **I** for several seconds, let **tMin = 0** and **tMax = 5**. The other values in the window editor should not affect the table. (Figure 3.17)

```

Plot1 Plot2 Plot3
\Q'1=Q2
\Q'2=1-2 Q1-2 Q2

```

Q'0= WIND INITC AXES GRAPH  
t 0 INSE DELF SELCT

Figure 3.15

```

INITIAL CONDITIONS
tMin=0
■ Q11=0
■ Q12=0
Q13=2

```

Q'0= WIND INITC AXES GRAPH

Figure 3.16

```

WINDOW
tMin=0
tMax=5
tStep=.001
tPlot=0
xMin=-5
xMax=5

```

Q'0= WIND INITC AXES GRAPH

Figure 3.17

4. Set up the table using TABLE SETUP (TBLST). (Figure 3.18)

Figure 3.18

5. Press  $\boxed{F1}$  (TABLE) to see the table. There will be a pause before the table appears while the TI-86 calculates the values in the table. (Figure 3.19)

| t   | Q1       | Q2       |
|-----|----------|----------|
| 0   | 0        | 0        |
| .5  | .0881583 | .2909486 |
| 1   | .2451292 | .3102779 |
| 1.5 | .3798938 | .2240022 |
| 2   | .4657714 | .1250554 |
| 2.5 | .5079035 | .0514297 |

Figure 3.19

The charge ( $Q1$ ) increases from 0 to .51 while the current ( $Q2$ ) briefly increases from 0 to .31 and then decreases to almost zero. Graphical and analytic solutions to this differential equation indicate a damped oscillation as the charge  $Q$  approaches its limit.

You can obtain the graphical solution by changing the viewing window and graphing. (Figures 3.20, 3.21, and 3.22)

Can you see connections between the graph in Figure 3.22 and the table in Figure 3.19?

Figure 3.20

Figure 3.21

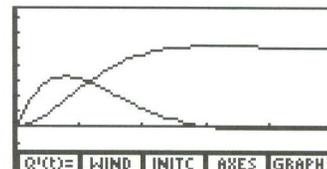


Figure 3.22

The analytic solutions for  $I$  and  $Q$  are:

$$I = e^{-t} \sin t.$$

$$Q = -\frac{1}{2}(e^{-t} \cos t + e^{-t} \sin t - 1).$$

You can use this method of converting higher order differential equations to systems of first order differential equations to solve the following exercises.

**Exercises**

1. Graph the solution to the differential equation

$$2y'' + y' + 2y = 0$$

with initial conditions

$$y(0) = 0; \quad y'(0) = 1.25.$$

Use **DrawF** to compare with the solution

$$y = 1.29099e^{-0.25x} \sin(0.968246x).$$

2. Graph the solution to the differential equation

$$y'' + y = \sec x$$

with initial conditions

$$x = 0, y = 0, y' = 0.$$

Use **DrawF** to compare with the solution

$$y = \ln|\cos x| \cos x + x \sin x.$$

**Hint:** Use a  $[0, 2\pi]$   $x$   $[-10, 5]$  viewing window with  $tMin = 0$ ,  $tMax = 2\pi$  and  $difTol = .1$ .

3. Graph the solution to

$$y''' + 3y'' - 2y' + y = 0$$

with initial conditions

$$t = 0, y = 0, y' = 1, y'' = 1.$$

(Change **difTol** back to .001.)

4. Graph the solution to

$$y'''' - (y'')^2 + 2y' = 0$$

with initial conditions

$$t = 0, y = 0, y' = 1, y'' = 1.$$

5. A projectile is launched straight up from an initial height of 2 meters with an initial velocity of 30 meters/second. The acceleration due to gravity is
- $-9.8$
- meters/sec
- <sup>2</sup>
- . How long will the projectile be in the air? How fast will it be moving when it hits the ground?
- 
6. Make a table of values for the charge on a capacitor and the current in an LRC circuit if

$$E = 9, R = 5, L = 2, \text{ and } C = \frac{1}{5}.$$

Assume when  $t = 0$ ,  $Q = 0$  and  $I = 0$ . How long will it take for the capacitor to charge up? What is the final charge on the capacitor?