EXPLORATIONS

Chapter 10

Features Used

right( ), product( ), ./, .*, list▸mat( ), mod( ), For...EndFor, norm( ), unitV( ), [∫], ans(1), Func, NewProb, comDenom( ), unitV( ), [d], [≈], If...Elseif...Endif

Setup

[◆] 1, setFold(emag)

Vector Calculus

This chapter describes how to use the TI-89 for differential and integral calculus. The differential operations illustrated include gradient, divergence, curl, and Laplacian. Line and surface integrals also are included. The functions **cordchk( ), grad( ), div( ), curl( ),** and **lap( )** are created in this chapter, and the functions **rec2sph( )** and **cyl2rec( )** from *Chapter 9: Vectors* are used.

Topic 45: Gradient

The gradient is a differential vector operation which gives the magnitude and direction of the greatest rate of change of a scalar potential. Calculation forms are usually given in rectangular, cylindrical, and spherical coordinates. However, the single form

$$\text{grad}\, v = \nabla v = \frac{\partial v}{h_1 \partial x_1}\mathbf{a}_1 + \frac{\partial v}{h_2 \partial x_2}\mathbf{a}_2 + \frac{\partial v}{h_3 \partial x_3}\mathbf{a}_3$$

can serve for all three coordinate systems where x_i is the ith variable and $\mathbf{a}_i$ is the unit vector associated with the ith variable. h_i is called the metric for the ith variable; it is multiplied by angular variables to calculate length in the angular direction. The table below shows these elements for the three coordinate systems.

	x_i	h_i	$\mathbf{a}_i$
Rectangular	x,y,z	1,1,1	$\mathbf{a}_x,\mathbf{a}_y,\mathbf{a}_z$
Cylindrical	ρ,φ,z	1,ρ,1	$\mathbf{a}_\rho,\mathbf{a}_\phi,\mathbf{a}_z$
Spherical	r,θ,φ	1,r,rsinθ	$\mathbf{a}_r,\mathbf{a}_\theta,\mathbf{a}_\phi$

Table 1. Variables, metrics, and unit vectors

The similar form for the gradient definition in the three coordinate systems means that the same instructions can be used for all of the coordinate systems. Once the coordinate system has been selected, the variables and metrics for that coordinate system can be applied to the derivatives of the potential.

These instructions are included in the function called **grad**. The arguments for this function are the desired coordinate system and the mathematical form of the potential. Specification of the coordinate system determines the variables and metrics used in **grad**. Since other vector operations also require variable and metric selection, a separate function, **cordchk**, is created for this.

1. Clear the TI-89 by pressing [2nd] [F6] **2:NewProb** [ENTER] .
2. Press [APPS] **7:Program Editor 3:New** and select **Type: Function**.
3. Name the **Variable: cordchk**.
4. Enter the instructions for **cordchk** listed below.

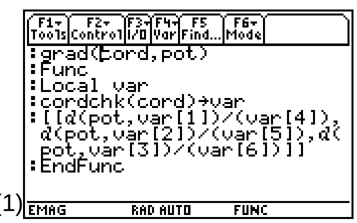
```
:cordchk(cord)
:Func
:If string(cord)="rec" Then
:  {x,y,z,1,1,1}
:ElseIf string(cord)="cyl" Then
:  {ρ,φ,z,1,ρ,1}
:ElseIf string(cord)="sph" Then
:  {r,θ,φ,1,r,r*sin(θ)}
:EndIf
:EndFunc
```

***Note**: To enter ρ, press [◆] [(] [alpha] **r**; to enter [θ], press [◆] [θ]; and to enter ϕ, press [◆] [(] [alpha] **f**.*

cordchk accepts coordinate arguments of **rec**, **cyl**, or **sph** from which it returns a list of the form $\{x_1,x_2,x_3,h_1,h_2,h_3\}$.

5. Create a new function named **grad**.
6. Enter the instructions for **grad** as shown in screen 1.

grad accepts the coordinate argument of **rec**, **cyl**, or **sph** and a symbolic form of the potential argument expressed in the variables of the chosen coordinate system. The results of the **cordchk** function are stored as a local variable, **var**. The elements of **var** are used to calculate the vector components of the gradient. The resulting vector represents the three components of the gradient in the coordinate system of the calculation. The order of the components is (x,y,z), (ρ,φ,z), or (r,θ,φ).

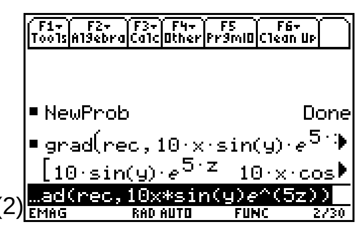
(1)


7. Return to the Home screen and calculate the gradient of $v=10x\sin(y)\exp(e^{-5z})$ as shown in screen 2.

grad [(] **rec** [,] **10x** [×] [2nd] [SIN] **y** [)] [◆] [e^x] [(-)] **5z** [)] [)]

The complete answer is

[10sin(y)E^{-5z} 10x cos(y)E^{-5z} –50x sin(y)E^{-5z}].

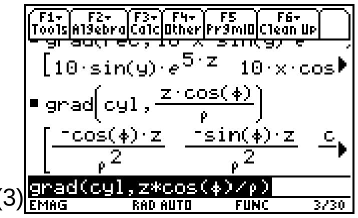
(2)


8. Find the gradient of **v=zcos(ϕ)/ρ** as shown in screen 3.

grad [(] **cyl** [,] **z** [×] [2nd] [COS] [◆] [(] [alpha] **f** [)] [÷] [◆] [(] [alpha] **r** [)]

The answer is

$$\begin{bmatrix} \frac{-\cos(\phi)z}{\rho^2} & \frac{-\sin(\phi)z}{\rho^2} & \frac{\cos(\phi)}{\rho} \end{bmatrix}$$

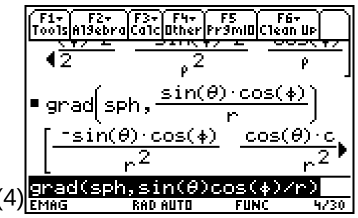
(3) 

9. Find the gradient of **v=sin(θ)cos(ϕ)/r** as shown in screen 4.

grad [(] **sph** [,] [2nd] [SIN] [◆] [θ] [)] [2nd] [COS] [◆] [(] [alpha] **f** [)] [÷] **r** [)]

The answer is

$$\begin{bmatrix} \frac{-\sin(\theta)\cos(\phi)}{r^2} & \frac{\cos(\theta)\cos(\phi)}{r^2} & \frac{-\sin(\phi)}{r^2} \end{bmatrix}$$

(4) 

Topic 46: Surface Normal

Since the gradient points in the direction of greatest rate of change of a function, it is perpendicular to a surface on which that function is constant. The unit normal vector to a surface can be found using this property as $\mathbf{a}_N=\nabla \mathbf{f}/|\nabla \mathbf{f}|$ where **f** is the function which describes the surface.

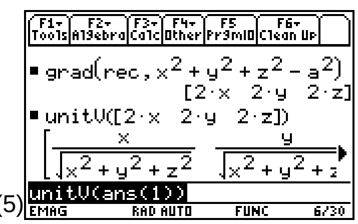
Find the unit normal vector to a sphere of radius **a**. The sphere is described by a function of $f=x^2+y^2+z^2-a^2$.

1. Use the function **grad** from Topic 45 to calculate the gradient.

grad(**rec,x^2+y^2+z^2-a^2**)

2. Use **unitV()** to find the unit normal vector (bottom of screen 5).

[CATALOG] **unitV(** [2nd] [ANS]

(5) 

The answer is

$$\begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{bmatrix}$$

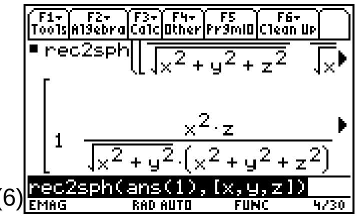
3. To verify that the unit normal vector to the spherical surface is radial, convert to spherical components using **rec2sph** from Topic 43 as shown in screen 6.

rec2sph [(] [2nd] [ANS] [,] [2nd] [[] **x** [,] **y** [,] **z** [2nd] []] [)]

The complete answer is

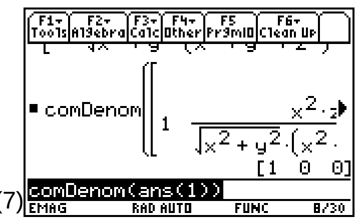
$$\begin{bmatrix} 1 & \dfrac{x^2 z}{\sqrt{x^2+y^2}\left(x^2+y^2+z^2\right)}+\dfrac{\dfrac{y^2 z}{\sqrt{x^2+y^2}}-\sqrt{x^2+y^2}\,z}{x^2+y^2+z^2} & 0 \end{bmatrix}$$

in which the terms correspond to the r-, θ-, and ϕ-components, respectively.

(6)


4. The second term, although rather complicated, has some common terms. Use **ComDenom()** to simplify it as shown in screen 7.

The TI-89 built-in rules of algebra show that the second term is zero. This agrees with intuition that the perpendicular to a sphere is radial only.

(7)


Topic 47: Divergence

Divergence of a flux density is the differential vector operation which indicates the net flux emanating from a point. When there are sources of flux at the point, the divergence is positive; when there are sinks, it is negative. The mathematical description of divergence is

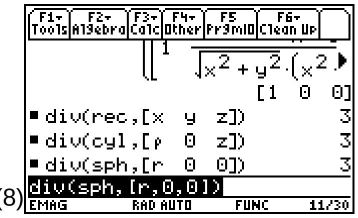
$$\text{div}\mathbf{D} = \nabla \bullet \mathbf{D} = \frac{\dfrac{\partial(D_1h_2h_3)}{\partial x_1}+\dfrac{\partial(D_2h_3h_1)}{\partial x_2}+\dfrac{\partial(D_3h_1h_2)}{\partial x_2}}{h_1h_2h_3}$$

where the D_i is the ith component of the flux density vector **D** and h_i and x_i are defined as before (Topic 45). These operations are defined in the function **div** shown below.

```
:div(cord,fld
:Func
:Local  var,met
:cordchk(cord)→var
:right(var,3)→met
:product(met)*fld ./(list▶mat(met))→fld
:(d(fld[1,1],var[1])+d(fld[1,2],v
 ar[2])+d(fld[1,3],var[3]))/(product(met))
:EndFunc
```

The function **cordchk** from Topic 45 determines the list of variables and metrics for the chosen coordinate system and stores them in local variable, **var**. The three metrics are extracted from **var** and stored in **met** using **right().** Scalar multiplication of the field by the product of the metrics dot divided (./) by the vector of the metrics forms the $D_ih_jh_k$ term. The partial derivatives and division by the metric product completes the calculations of **div**.

The position vector from the origin to a point in space is expressed in the three coordinate systems as [x,y,z], [ρ,0,z], and [r,0,0]. The divergence of the position vector should be the same in all three coordinate systems since the coordinate system should not alter the properties of the vector. The calculations shown in screen 8 verify this.



(8)

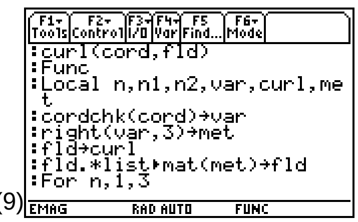
Topic 48: Curl

The curl of a vector field is a measure of its vorticity, which is its tendency to rotate about a point. When the curl of a field is zero everywhere, it is known as a conservative field and an integral around any closed path is zero. The electrostatic field is conservative; the magnetostatic field is not conservative. The mathematical definition of the curl is

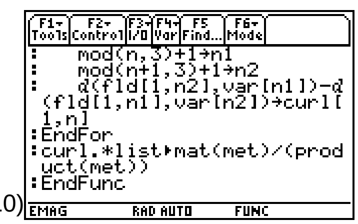
$$\text{curl}\mathbf{H} = \nabla \times \mathbf{H} = \frac{\sum_{i=1}^{3}\left[\frac{\partial(h_k H_k)}{\partial x_j} - \frac{\partial(h_j H_j)}{\partial x_k}\right] h_i \mathbf{a}_i}{h_1 h_2 h_3}$$

where x_i, $\mathbf{a}_i$, h_i, and H_i are the variable, unit vector, metric, and vector component of the ith coordinate. (i,j,k) form a right-handed system. Although the form of the curl is somewhat more complicated than previous vector operations, its cyclic nature makes it easy to implement.

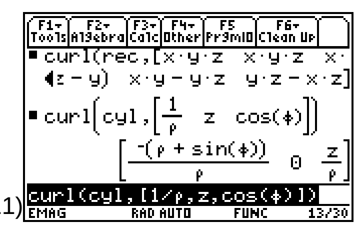
The calculations of the curl are implemented in the function **curl**.



(9)



(10)



(11)

1. Define the function **curl** as shown in screens 9 and 10. Notice that it includes the function **cordchk** from Topic 45.

   The variables and metrics are stored in local variables **var** and **met**; the elements of **fld** serve as dummy elements to form the local variable **curl**. **n**, **n1**, and **n2** form a cyclical triad used to compute the derivatives of **fld**. Each pass through the **For** loop forms one of the vector components which replaces the dummy elements stored in **curl**.

2. Return to the Home screen, and enter the curl of [xyz,xyz,xyz] as shown in the top of screen 11.

   curl [(] **rec** [,] [2nd] [[] **x** [×] **y** [×] **z** [,] **x** [×] **y** [×] **z** [,] **x** [×] **y** [×] **z** [2nd] []] [)]

   The answer is [x(z-y) xy-yz yz-xz], as shown at the top of screen 11.

3. Find the curl of [1/ρ,z,cosϕ] as shown in the bottom of screen 11.

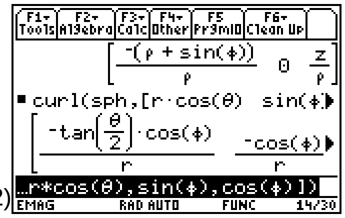
   curl [(] **cyl** [,] [2nd] [[] **1** [÷] [◆] [(] [alpha] **r** [,] **z** [,] [2nd] [COS] [◆] [(] [alpha] **f** [)] [2nd] []] [)]

4. Find the curl of [rcosθ, sinφ, cosφ] as shown in screen 12.

curl [(] **sph** [,] [2nd] [[] **r** [×] [2nd] [COS] [◆] [θ] [)] [,] [2nd] [SIN] [◆] [(] [alpha] **f** [)] [,] [2nd] [COS] [◆] [(] [alpha] **f** [)] [2nd] []] [)]

The answer is

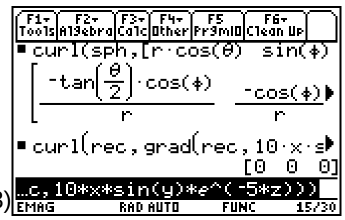
$$\begin{bmatrix} \dfrac{-\tan\left(\dfrac{\theta}{2}\right)\cos(\phi)}{r} & \dfrac{-\cos(\phi)}{r} & \dfrac{\sin(\theta)r+\sin(\phi)}{r} \end{bmatrix}$$



(12)

5. A vector theorem states that the curl of the gradient of a potential is identically zero. The example of a gradient in rectangular coordinates from Topic 45 demonstrates this as **curl(rec,grad(rec,10xsin(y)e^{-5z}))=0** (screen 13).

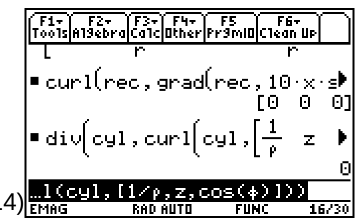
curl [(] **rec** [,] **grad** [(] **rec** [,] **10x** [×] [2nd] [SIN] **y** [)] [×] [◆] [e^x] [(-)] **5z** [)] [)] [)]



(13)

6. Another vector theorem states that the divergence of any curl is identically zero. This is demonstrated by **div(cyl,curl(cyl,[1/ρ,z,cos(φ)]))=0** (screen 14).

div [(] **cyl** [,] **curl** [(] **cyl** [,] [2nd] [[] **1** [÷] [◆] [(] [alpha] **r** [,] **z** [,] [2nd] [COS] [◆] [(] [alpha] **f** [)] [2nd] []] [)] [)]



(14)

Topic 49: Laplacian

The behavior of many physical potentials is mathematically described in rectangular coordinates by

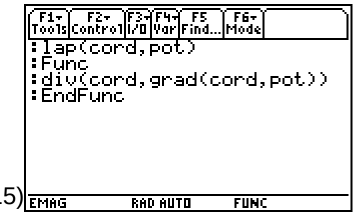
$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}$$

Since this often occurs in Laplace's equation, this is known as the Laplacian. An alternate form of the Laplacian is

$$\text{lapv}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\nabla\bullet\nabla v=\nabla^2 v$$

From the $\nabla\bullet\nabla v$ term, the method of forming the Laplacian is obvious; it is the divergence of the gradient of the potential v. Although vectors and vector operations are involved, the Laplacian produces a scalar result.

1. Define the function **lap** as shown in screen 15. Notice that it uses the function **div** from Topic 47 and the function **grad** from Topic 45.



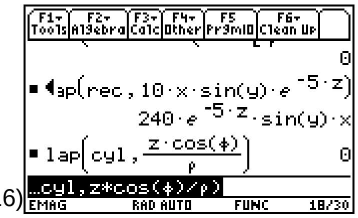
(15)

2. Return to the Home screen and find the Laplacian of v=10xsin(y)e^{-5z} as shown on the top of screen 16.

lap [(] **rec** [,] **10x** [×] [2nd] [SIN] **y** [)] [×] [◆] [e^x] [(-)] **5z** [)] [)]

3. Find the Laplacian of v=zcos(ϕ)/ρas shown on the bottom of screen 16.

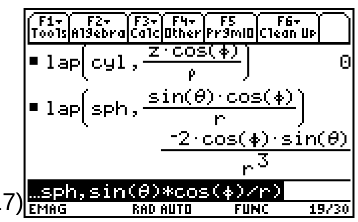
lap [(] **cyl** [,] **z** [×] [2nd] [COS] [◆] [(] [alpha] **f** [)] [÷] [◆] [(] [alpha] **r** [)]



(16)

4. Find the Laplacian of v=sin(θ)cos(ϕ)/r as shown on screen 17.

lap [(] **sph** [,] [2nd] [SIN] [◆] [θ] [)] [×] [2nd] [COS] [◆] [(] [alpha] **f** [)] [÷] **r** [)]



(17)

Topic 50: Line Integrals

The potential difference, also called voltage drop, between two points is given in the form of a line integral as

$$V_A - V_B = -\int_{\text{PathB}\to\text{A}} \mathbf{E}\bullet\mathbf{dl}$$

In electrostatic cases, this integral is independent of the path, that is, any path between the two endpoints is valid. In general, however, the integral must be evaluated along a specific path. Evaluation of line integrals along a path can be simplified to three steps:

1. Evaluate the dot product of the integrand in the most convenient coordinate system.
2. Include the effects of the path.
3. Perform the resulting scalar integrals.

Example 1

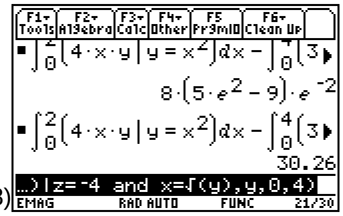
Evaluate

$$\int_{\text{Path}} \mathbf{F}\bullet\mathbf{dl}$$

where $\mathbf{F}=4xy\mathbf{a}_x-3ze^{-x}\mathbf{a}_y$ along the parabolic path $y=x^2$ in the z=-4 plane from (0,0,-4) to (2,4,-4). Since the path and the variables are specified in rectangular coordinates, the form of the differential is chosen as rectangular also.

Step 1 leads to $\mathbf{F}\bullet\mathbf{dl}=4xydx-3ze^{-x}dy$.

Step 2 requires that along the path, $y=x^2$ and z=-4. These substitutions give the first term as $4xydx=4x(x^2)\,dx$, and the second term as $-3ze^{-x}dy = -3(-4)e^{-\sqrt{y}}\,dy$. These explicit substitutions introduce the effects of the path into the integrand. The substitutions have been chosen so that the variables of each integrand are the same as the differential. These substitutions are simplified using the "with" operator, [|].



(18)

1. Now do step 3 on the TI-89 to evaluate the resulting integrals.

   [2nd] [∫] **4x** [×] **y** [|] **y** [=] **x** [^] **2** [,] **x** [,] **0** [,] **2** [)] [−] [2nd] [∫] **3 z** [◆] [e^x] [(-)] **x** [)] [|] **z** [=] [(-)] **4** [CATALOG] **and x** [=] [2nd] [√] **y** [)] [,] **y** [,] **0** [,] **4** [)]

   The exact value result is $8(5e^2-9)e^{-2}$ (top of screen 18).

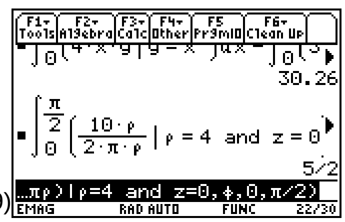
2. Pressing [◆] [≈] gives 30.26 in floating point form (bottom of screen 18).

Example 2

Evaluate

$$\int_{\text{Path}} \mathbf{H} \bullet \mathbf{dl}$$

where $\mathbf{H}=(10/2\pi\rho)\mathbf{a}_\phi$ along the path ρ=4, z=0, and $0\le\phi\le\pi/2$. The natural coordinate system is cylindrical so $\mathbf{H}\bullet\mathbf{dl}=10\rho d\phi/2\pi\rho$. Note that dϕ is multiplied by the metric ρ to get the differential length in the ϕ direction, ρdϕ.

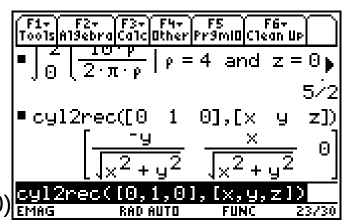


(19)

1. The integral is entered as shown in screen 19.

   [2nd] [∫] **10** [◆] [(] [alpha] **r** [÷] [(] **2** [2nd] [π] [◆] [(] [alpha] **r** [)] [|] [◆] [(] [alpha] **r** [=] **4** [CATALOG] **and z** [=] **0** [,] [◆] [(] [alpha] **f** [,] **0** [,] [2nd] [π] [÷] **2** [)]

   The result is 5/2.



(20)

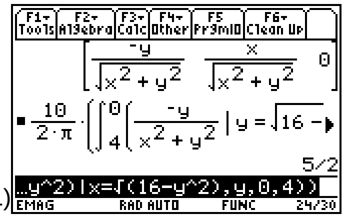
2. For those not as comfortable with cylindrical coordinate integration, convert the problem to rectangular coordinates. First, transform $\mathbf{a}_\phi$ to rectangular coordinates using the **cyl2rec** function from Topic 42 (Chapter 9) to obtain $\mathbf{a}_\phi = -y/\sqrt{(x^2+y^2)}\mathbf{a}_X + x/\sqrt{(x^2+y^2)}\mathbf{a}_Y$ (screen 20).

   With the relationship $\rho = \sqrt{x^2+y^2}$, the dot product of the integrand becomes

   $$\mathbf{H}\bullet\mathbf{dl} = \frac{10}{2\pi\sqrt{x^2+y^2}}\left[\frac{-ydx}{\sqrt{x^2+y^2}} + \frac{xdy}{\sqrt{x^2+y^2}}\right]$$

   The integration follows the circular path on which $x^2+y^2=16$.

This is entered as shown in screen 21.


(21)

10 [÷] [(] **2** [2nd] [π] [)] [(] [2nd] [∫] [(-)] **y** [÷] [(] **x** [^] **2** [+] **y** [^] **2** [)] [|] **y** [=] [2nd] [√] **16** [-] **x** [^] **2** [)] [,] **x** [,] **4** [,] **0** [)] + [2nd] [∫] **x** [÷] [(] **x** [^] **2** [+] **y** [^] **2** [)] [|] **x** [=] [2nd] [√] **16** [-] **y** [^] **2** [)] [,] **y** [,] **0** [,] **4** [)] [)]

The result is 5/2 as before. Note that **x** goes from 4 to 0 and **y** from 0 to 4 as ϕ varies from 0 to π/2.

Topic 51: Surface Integrals

The flux passing through a surface is expressed by surface integrals such as

$$\iint_{\text{Area}} \mathbf{J} \bullet \mathbf{ds}$$

where **J** is the flux density and **ds** is the directed surface element in a specified direction. The evaluation of surface integrals is similar to the three-step process used with line integrals in Topic 50:

1. Evaluate the integrand.
2. Include the effects of the surface.
3. Evaluate the resulting integrals.

Two examples follow which illustrate this process.

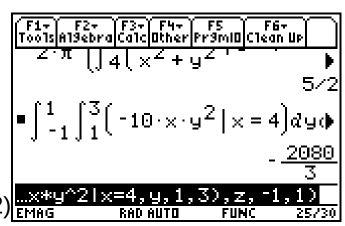
Example 1

Find the flux

$$\iint_{\text{Area}} \mathbf{J} \bullet \mathbf{ds}$$

passing through the x=4 surface in the $-\mathbf{a}_x$ direction for which $1 \le y \le 3$ and $-1 \le z \le 1$ where $\mathbf{J} = 10xy^2\mathbf{a}_x$. The integrand when evaluated on the surface is given by $\mathbf{J}\bullet\mathbf{ds} = {}^-10xy^2dydz|_{x=4} = {}^-40y^2dydz$.

The resulting integral is entered as shown in screen 22.


(22)

[2nd] [∫] [2nd] [∫] [(-)] **10x** [×] **y** [^] **2** [|] **x** [=] **4** [,] **y** [,] **1** [,] **3** [)] [,] **z** [,] [(-)] **1** [,] **1** [)]

The result is $^-2080/3$.

Example 2

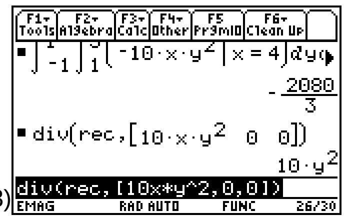
The divergence theorem states that the flux out of a closed surface integral is equal to the divergence of the flux density throughout the volume

$$\oint\!\!\oint_A \mathbf{J} \bullet \mathbf{ds} = \iiint_V \nabla \bullet \mathbf{J} dv$$

Calculate the flux out of the entire unit sphere due to $\mathbf{J}=10xy^2\mathbf{a}_x$ using the function **div** from Topic 47.

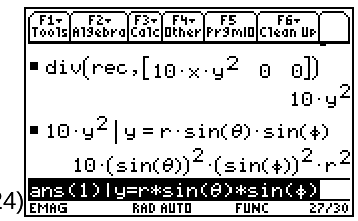
1. Calculate div$\mathbf{J}=10y^2$ (screen 23).

div [(] **rec** [,] [2nd] [[] **10x** [×] **y** [^] **2** [,] **0** [,] **0** [2nd] []] [)]

(23) 

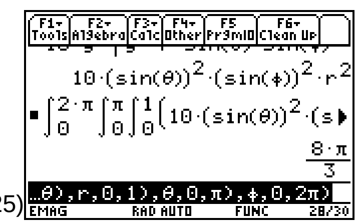
2. Integrate this result throughout the volume of the sphere. Setting the limits of integration with respect to rectangular coordinates is tedious because the volume naturally fits spherical coordinates. Instead, transform the single variable of the integrand to y=rsinθsinϕ (screen 24).

[2nd] [ANS] [|] **y** [=] **r** [×] [2nd] [SIN] [◆] [θ] [)] [×] [2nd] [SIN] [◆] [(] [alpha] **f** [)]

(24) 

3. Integrate throughout the volume of the sphere using the differential volume of spherical coordinates $dv=r^2\sin\theta drd\theta d\phi$ to obtain the total flux of 8π/3 as shown in screen 25.

[2nd] [∫] [2nd] [∫] [2nd] [∫] [2nd] [ANS] [×] **r** [^] **2** [×] [2nd] [SIN] [◆] [θ] [)] [,] **r** [,] **0** [,] **1** [)] [,] [◆] [θ] [,] **0** [,] [2nd] [π] [)] [,] [◆] [(] [alpha] **f** [,] **0** [,] **2** [2nd] [π] [)]

(25) 

Tips and Generalizations

WOW! Triple integrals on a pocket calculator, and this is just a warm up. This chapter showed how powerful vector calculus operations can be performed by defining a few simple functions (**grad()**, **div()**, **curl()**, **lap()**). These combined with **with** ([|]) provide a convenient way to do vector calculus.

Chapter 11 shows how these operations can be used to solve typical electromagnetics problems.