# EXPLORATION 5 The Quest for Roots of Higher Order Equations

The study of algebra began as the quest for roots of equations. Cuneiform clay documents of the ancient Babylonians (c. 1700 B.C.) reveal that they were able to solve linear and quadratic equations expressed in rhetorical style. When algebraic symbolism was developed during the Italian Renaissance in the 16th and 17th centuries, those procedures for finding roots of linear and quadratic equations emerged as the familiar formulas that we studied in the previous explorations.

As early as 1100 A.D. Omar Khayyam, the Persian poet and mathematician, discovered a geometric procedure for finding the real roots of cubic equations. However, it was not until the 16th century that an algebraic formula was found for roots of the general cubic equation $ax^3 + bx^2 + cx + d = 0$. The origin of this discovery remains one of the great enigmas in the history of mathematics. The following sequence offers a brief overview of the events leading to the ultimate conquest in the search for roots of algebraic equations.

**1**

•1515 Scipioni del Ferro discovers how to solve equations of the form $x^3 + cx = d$. He reveals this discovery in secrecy to his student, Antonio Maria Fior.

•1535 Niccolo Tartaglia claims to have discovered how to solve equations of the form $x^3 + bx^2 = d$, so Fior challenges him to a public equation-solving duel.

•Tartaglia wins the duel having discovered only days earlier how to solve equations of the type known to Fior.

•Tartaglia confides his method to Professor Girolamo Cardano.

Nicolo Tartaglia 1499-1557

**2**

•1545 Girolamo Cardano publishes *Ars magna* in which he presents the formula for the roots of the general cubic equation, violating his pledge of secrecy to Tartaglia.

•Tartaglia protests vehemently, accusing Cardano of plagiarism. Ludovico Ferrari, a brilliant student of Cardano, defends his teacher, claiming that Cardano's source was Fior!

•*Ars magna* included also the solution to the general quartic equation discovered by that same Ferrari in 1540 when he was still a teenager!

Girolamo Cardano 1501-1576

**3**

•1750 Leonhard Euler fails to solve the general quintic equation, $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$.

•1824 Neils Abel proves that the roots of the general quintic equation cannot be expressed by means of radicals involving only the coefficients of the equation.

•1832 Evariste Galois dies at the age of 21 in a pistol duel over a young lady. His legacy of 63 pages, many of them frantically scrawled the night before the duel, contained for an algebraic equation of any degree necessary and sufficient conditions that it be solvable by radicals.

Evariste Galois 1811-1832

## WORKED EXAMPLE 1

An ancient Babylonian tablet displays a table of the values of $n^3 + n^2$ for $n = 1$ through 30. Create such a table to find a root of the equation:

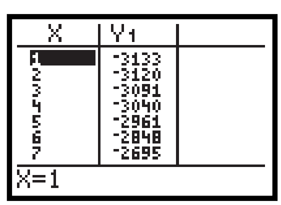
$$x^3 + 2x^2 - 3136 = 0$$

### Solution

To create the required table, we press:

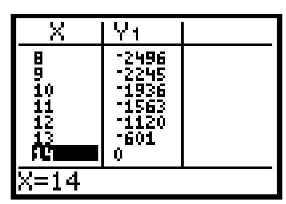
[Y =] [X,T,θ,n] [^] [3] [+] [2] [X,T,θ,n] [x²] [–] [3] [1] [3] [6] [2nd] [TBLSET] [1] [ENTER] [1] [2nd] [TABLE]

The display shows the table below left.



| X | Y1 |
|---|---|
| 1 | -3133 |
| 2 | -3120 |
| 3 | -3091 |
| 4 | -3040 |
| 5 | -2961 |
| 6 | -2848 |
| 7 | -2695 |

X=1



| X | Y1 |
|---|---|
| 8 | -2496 |
| 9 | -2245 |
| 10 | -1936 |
| 11 | -1563 |
| 12 | -1120 |
| 13 | -601 |
| 14 | 0 |

X=14

Using the [▼] key, we scroll down to X = 14 where we observe $Y_1 = 0$, so 14 is a root of the equation.

## WORKED EXAMPLE 2

Graph the curve defined by the equation:

$$y = 2x^3 - 6x^2 - x + 6$$

Then trace along the curve to find the approximate values of the roots of the corresponding equation.

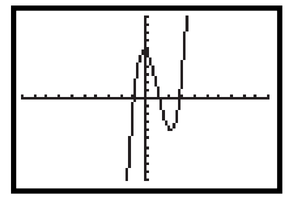
$$2x^3 - 6x^2 - x + 6 = 0$$

**Solution**

To graph this equation, we press:

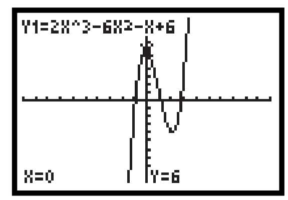
[Y =] [2] [X,T,θ,*n*] [^] [3] [–] [6] [X,T,θ,*n*] [$x^2$] [–] [X,T,θ,*n*] [+] [6] [ENTER] [GRAPH]

The screen displays the graph in the display.

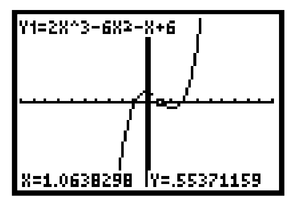


We observe that the graph crosses the $x$-axis three times, indicating that there are three real values of $x$ for which $y$ is zero.

To approximate these roots, we press [TRACE] and observe a flashing cursor at the top of the "hill" or maximum of the curve. The screen indicates that the cursor is at the point on the curve with coordinates (0, 6).

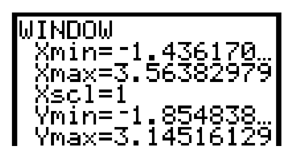



By pressing the cursor keys [◄] and [►], we can trace along the curve to the points where the curve crosses the $x$-axis. If we trace along the curve to the right, we reach the point (1.063…, .553…). The next point in the tracing is (1.276…, -.893…). Since the y-coordinate changes sign, we conclude that the curve crosses the $x$-axis somewhere between these two points. That is, if $x_1$ denotes the root, then $1.063 \leq x_1 \leq 1.276…$.

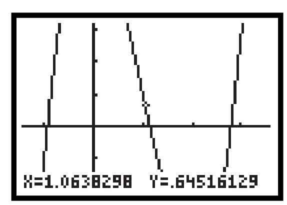



To obtain a closer approximation to this root, trace along the curve to the point (1.063…, .553…) and then press:

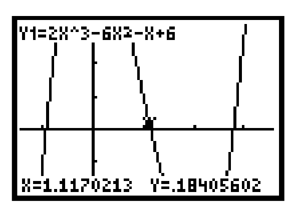
[ZOOM] [2] [ENTER]




This keying sequence is called *zooming in* and it causes the curve to be redrawn with the window shrunk by a factor of 4;
i.e. $-1.43... \leq x \leq 3.56...$, $-1.85... \leq y \leq 3.14...$ (See the display).




Proceeding as above, we obtain $1.1170.. \leq x_1 \leq 1.1702…$.
Repeated zooming yields closer approximations to $x_1$.
For approximations to the roots $x_2$ and $x_3$ see *exercise* 5.



## *WORKED EXAMPLE 3*

Determine the roots of the equation

$$2x^3 - 6x^2 - x + 6 = 0$$

correct to 6 decimal places.

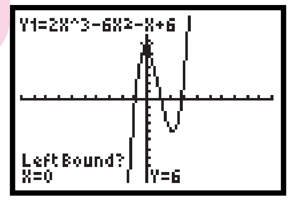
**Solution**

We graph this equation using the keying sequence in *worked example* 2. It is clear from the graph that this equation has three real roots.

To determine the value of the smallest positive root, $x_1$, without using the repeated iterations in *worked example* 2, we press these keys:

**2nd** [CALC] **2**

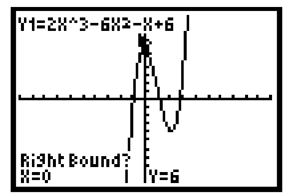
**Note:** A left bound is any value of $x$ which is less than $x_1$, so we could trace along the curve to any point left of ($x_1$, 0) and press **ENTER** .




We obtain the display shown.

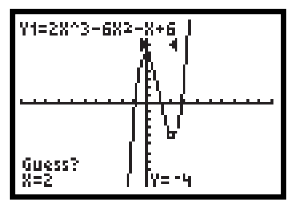
The prompt, **Left Bound?** requests that we enter a value of $x$ which is left of the root, $x_1$,(i.e. $x < x_1$). Since $0 < x_1$, we press:

**0** **ENTER**



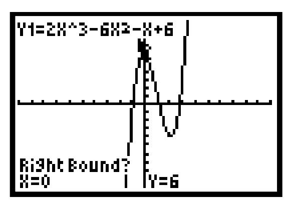

The display shows the prompt, **Right Bound?**. The scale markings on the $x$-axis show that $x_1 < 2$, i.e. 2 is on the right of $x_1$, so we press:

**2** **ENTER**




The display shows the prompt, **Guess?**. To enter this upper bound as our guess, we press the **ENTER** key one more time. The screen shows the cursor on the graph near the root and displays:

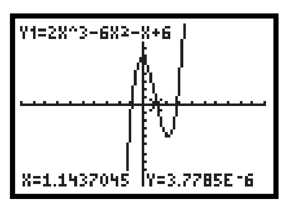
**Zero**
**X = 1.143705   Y = 0**




This indicates that when $x = 1.143705$, the value of the polynomial is close to zero. To verify that this value of $x$ is an approximation to the actual root $x_1$ correct to 6 decimal places, we must verify that $1.1437045 \le x_1 < 1.1437055$. That is, we must show that the cubic polynomial changes sign (has a zero) in this interval.

To evaluate the polynomial at $x = 1.1437045$, we press:

**2nd** [CALC] **1**




In response to the prompt, **Eval X =** , we enter 1.1437045. The display shows the corresponding value of y is 3.7785 E -6.

Repeating this procedure, we find the value of y corresponding to 1.1437055 is -3.098 E -6. Since y changes sign in this interval, $x_1$ lies in this interval and $x_1 = 1.143705$ (to 6 decimal places).

Similarly, we determine $x_2$ and $x_3$. (See *exercise* 5.)

## *Exercises*

**1.** Graph each of the equations below. Then use the TRACE and the cursor keys to find approximations to *all* the (real) roots of each equation.

a) $y = 3x^2 - 5$ b) $y = 3x^2 + 2$

c) $y = 2x^2 - 5x - 3$ d) $y = 2x^2 - 5x + 5$

Use the [CALC] menu to check your answers. What did you discover in parts "b" and "d". Explain what happened.

**2.** Compare the number of real roots of equation $3x^2 - 5 = 0$ with the number of real roots of the equation $3x^2 + 2 = 0$. Compare the number of real roots of equations $2x^2 - 5x - 3 = 0$ and $2x^2 - 5x + 5 = 0$.

Explain how adding a constant to one side of an equation changes the graph and possibly the number of real roots of the resulting equation.

**3.** Graph each of the following functions in the window: $-5 \le x \le 5$; $-50 \le y \le 50$.

a) $y = 6x^4 - 5x^3 - 18x^2 + 10x + 12$

b) $y = -6x^5 + 5x^4 + 18x^3 - 10x^2 - 12x$

c) $y = 21x^3 + 19x^2 - 61x + 21$

d) $y = -6x^4 + 11x^3 + 13x^2 - 16x - 12$

**4.** Use the TRACE and the cursor keys to find the approximate roots of the corresponding equations in exercise **3**. (That is, find for each equation the approximate value of $x$ corresponding to $y = 0$.)

**5.** Approximate *all* real roots of each of the following equations to six decimal places.

a) $x^3 + x^2 - 3x - 1 = 0$

b) $2x^3 - 6x^2 - x + 6 = 0$

c) $\frac{1}{8}x^3 - x^2 + \frac{3}{2}x + 1 = 0$

d) $\frac{1}{2}x^4 - \frac{1}{2}x^3 - 5x^2 + 4x + 8 = 0$

Verify your answers as in *worked example* 3.

**6.** Write the equation of a function that has each set of zeros. Then graph that function in the window:
$-5 \le x \le 5$; $-50 \le y \le 50$.

a) $\{(-2, 0), (2, 0), (-1, 0)\}$
b) $\{(1, 0), (3, 0), (-\sqrt{2}, 0), (\sqrt{2}, 0)\}$
c) $\{(1 + \sqrt{3}, 0), (1 - \sqrt{3}, 0), (3, 0), (0, 0)\}$

## *Investigations*

**7.** From your discoveries in *exercise* 5, conjecture answers to the following. (An equation of *degree n* is an equation in which the highest power of $x$ is $x^n$.)

a) What is the maximum number of real roots which an equation of degree $n$ can have?
b) What is the minimum number of real roots which an equation of degree $n$ can have?
c) Is there a cubic equation that has no real roots? If so, give an example.

**8.** a) Graph the equation $y = \frac{1}{x^2} - 0.001$.

Use the TRACE key to find a value of $x$ for which $y$ is approximately 0. Trace along the curve to the right to find a value of $x$ for which $y < 0.001$. Denote this value by $x_0$. Is $x_0$ close to the actual root of this equation?

b) Use the [CALC] menu as in *worked example* 3 to find a positive root of the given equation. Is $x_0$ close in value to the positive root?

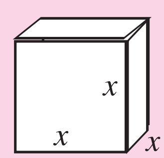
c) If $y$ is very close to 0 for a given value of $x$, is that value of $x$ a close approximation to a root of the equation? Explain your answer using an example.

d) Solve the given equation algebraically. Compare the root found algebraically with the root found using the [CALC] menu. Why do we not solve all equations algebraically?

### CHALLENGE

An ancient manuscript asks the edge length of a cube such that half its volume plus one sixth of its surface area is 4 units more than half the total length of all its edges. (Assume the edge length is $x$ units and that the surface area and volume are measured in squared and cubed units respectively.)

a) Write an algebraic expression for y in terms of $x$ if y is the difference between "half the volume plus one-sixth the surface area" and "half the total length of all its edges plus 4".




b) Graph the equation which expresses y in terms of $x$.

c) Approximate the real root to 2 decimal places.