Ages 15-17 – Exploring the binomial theorem

This nice exploration can be done using either DERIVE or one of the TI-CAS family. This is the TI-worksheet. It can easily be adapted for other Computer Algebra systems. You will find very few system-specific notes. The students know how to handle the very few CAS commands given in the worksheet.

In this activity we want to calculate - or **expand** - higher powers of binomials, such as $(4x^2 y - 3xy^3)^8$ How can we do this with your CAS?

First enter the expression, then simply "`expand`" it.

The result is impressive! What is the calculation time? ………………

Write down the 1st term of the expansion: ....

Write down the 4th term: .... and the last term: ....

Now use your system to expand the powers of $(a + b)$ and note only the coefficients of each term. (Write down the numbers neatly, one beneath the other.)

$n = 4$: $(a + b)^4$:

$n = 5$: $(a + b)^5$:

$n = 6$: $(a + b)^6$:

$n = 7$: $(a + b)^7$:

If you can recognize a pattern in your results, then add two more lines (for $n = 8$ and 9) without using your CAS. Then check the coefficients by using the CAS. If you do not see a pattern yet, then don´t worry, let your device do the work.

Complete the pattern for $n = 3, 2, 1$

How should the first line (for $n = 0$) look? ....

What can you conclude from this line? Check it!

....

Write down the numbers from the coefficients´ pattern in the form of a triangle:

($n = 0$)					1				
($n = 1$)					1	1			
($n = 2$)				1	2	1			
($n = 3$)									
($n = 4$)									
($n = 5$)									
($n = 6$)									
($n = 7$)									
($n = 8$)									
($n = 9$)									
($n = 10$)									
($n = 11$)									

Can you explain an obvious pattern?

We call this triangle of numbers **Pascal's Triangle** (Blaise Pascal, 1623-1662; but the triangle on the left is found in a Chinese paper from 1303 a.d.).
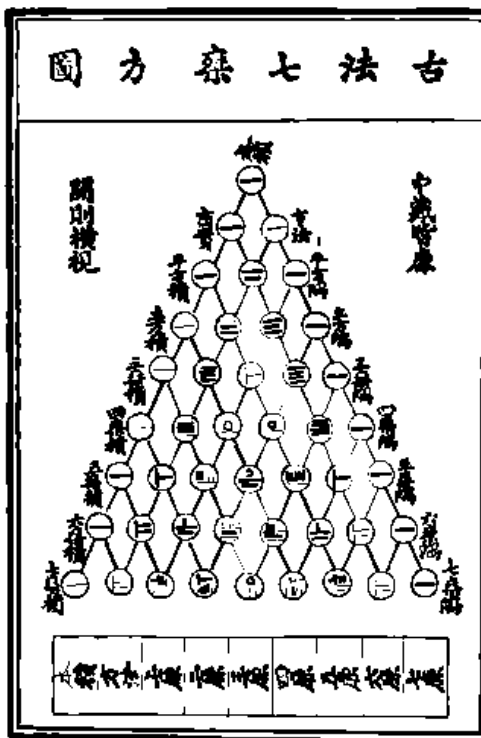
The other picture is from an old 16th century German book for merchants.
Both pictures are from the - highly recommended - book AGNESI to ZERO, Key Curriculum Press.

Now we are going to investigate some properties of the triangle.

Find the sum of the numbers in each line! (You can note the sum on the end of the lines.)

What do you notice? ....

Are you able to prove this fact?   (Hint: Substitute for a and b!)

....

The students should know the "coefficient proof" for quick checking of their results for calculating with general expressions, e.g. $(3a^2 + 4ab)(ab - 2a + 2b^2) = $ *........, by comparison of* $(3+4)(1-2+2) = 7$ *with the sum of the coefficients of the result.*

Observe the powers of a and b in the partial expressions of the expansions of the first five powers of $(a + b)$.

Try to expand $(a + b)^8$ by-hand using PASCAL´s Triangle.

$(a + b)^8 = $ ....

Check your result using the device.

Try once more with $(2x + 3y)^5$ without using CAS.

$(2x + 3y)^5 = $ ....

Expand all the partial expressions by-hand:

....

Check your results using CAS:

Expand $(a + b)^5$, then replace (substitute) a and b by the expressions $2x$ and $3y$ using the $[|]$-operator:

Compare the result offered by the CAS with your one!

Expanding once more if necessary will lead to a final result.

You can now expand $(2x + 3y)^5$ directly without intermediate substitution.

Now try by-hand: $(4z + 2u)^6 = $ ....

$= $ ....

Compare your result with the CAS-result!

Explain the effect of the changed sign in $(a - b)^n$ in your own words.

....

Exercises

Write down the following expansion without the machine and then check your results:

$(z + 4)^6$ = ...

= ....

Repeat the of exponents first (product and power of powers) and then proceed with the next problem.

$(4x^2y - 3xy^3)^8$ = ...

= ....

$(5ab + 2b^2)^9$ the 3rd term: ....   the 7^{tth} term: ....

$(2u + 1.5a)^5$ = ...

Expand the expression $(3a^2 - 2ab + b^2)^5$ without using the **expand**-command of your Computer Algebra System.

Describe how you tackled this problem.

....

Imagine that you have written down the result of the expansion, sorted with respect to descending powers of variable a. What are the first three terms? ....

Compare these with the result from the TI .

The rule or law for calculating the powers of binomials is called **the Binomial Law**.

The numbers appearing in Pascal´s triangle and their special order have fascinated mathematicians for a very long time, and they still do. Use the sheet on the next page to experiment with the numbers in this famous triangle.

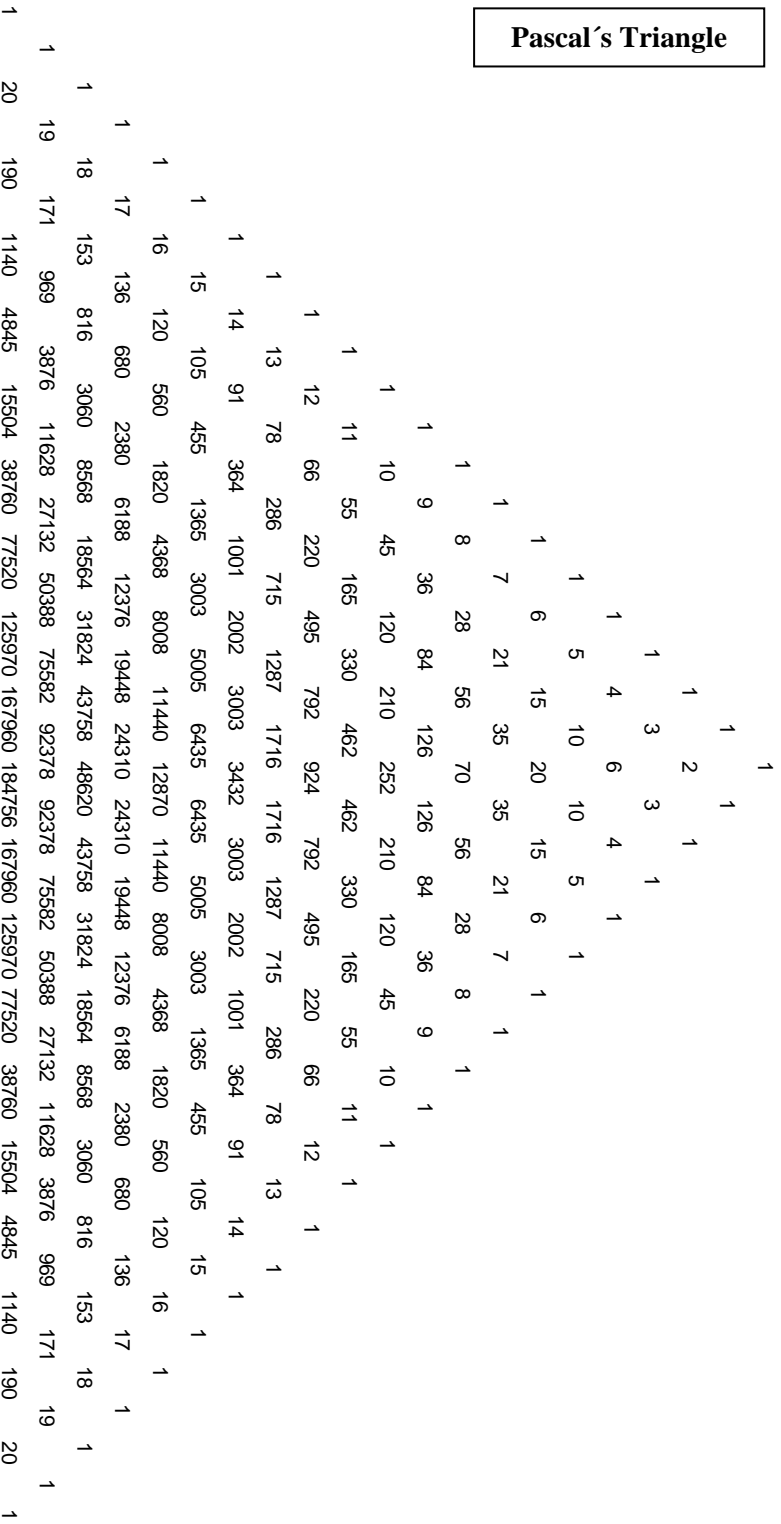Mark all even numbers using a felt pen. First make one or two copies for suggested investigations.

Do you observe a special pattern?

Suggested investigations:

a) Mark all numbers which are divisible by 3

b) Mark all numbers which leave the remainder 2 after division by 4

c) Design your own investigation of a similar kind.

We can recognize so called "self similarity" which is a property investigated in the very modern "Chaostheory".

Pascal´s Triangle

```
                                                    1
                                                 1     1
                                              1     2     1
                                           1     3     3     1
                                        1     4     6     4     1
                                     1     5    10    10     5     1
                                  1     6    15    20    15     6     1
                               1     7    21    35    35    21     7     1
                            1     8    28    56    70    56    28     8     1
                         1     9    36    84   126   126    84    36     9     1
                      1    10    45   120   210   252   210   120    45    10     1
                   1    11    55   165   330   462   462   330   165    55    11     1
                1    12    66   220   495   792   924   792   495   220    66    12     1
             1    13    78   286   715  1287  1716  1716  1287   715   286    78    13     1
          1    14    91   364  1001  2002  3003  3432  3003  2002  1001   364    91    14     1
       1    15   105   455  1365  3003  5005  6435  6435  5005  3003  1365   455   105    15     1
    1    16   120   560  1820  4368  8008 11440 12870 11440  8008  4368  1820   560   120    16     1
 1   17   136   680  2380  6188 12376 19448 24310 24310 19448 12376  6188  2380   680   136    17     1
1  18  153  816 3060 8568 18564 31824 43758 48620 43758 31824 18564 8568 3060  816  153  18   1
1 19 171 969 3876 11628 27132 50388 75582 92378 92378 75582 50388 27132 11628 3876 969 171 19 1
1 20 190 1140 4845 15504 38760 77520 125970 167960 184756 167960 125970 77520 38760 15504 4845 1140 190 20 1
```

Using the two programs `pas()` and `invpas()` you can generate various "pictures" emerging from Pascal´s Triangle.

`pas(n,divisor,remainder)` marks all locations in the first *n* rows of Pascal´s Triangle. where the division of the number by `divisor` leaves the `remainder`.

`invpas(n,divisor,remainder)` returns a black spot at these positions, where we do **not** receive the `remainder`.

To mark the even coefficients in the first 30 rows then one has to enter `pas(30,2,0)`. Explain the function arguments.

Try to reproduce your hand made paintings on the TI screen. How many rows can be represented on this small screen?

Which parameters in `pas()` generate the following pictures?
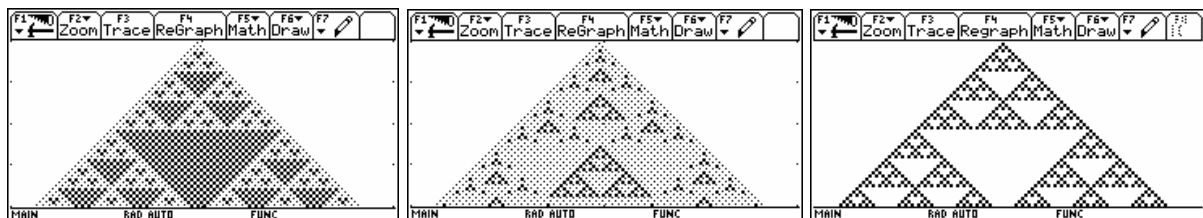
Try to produce two "extra nice" Pascal pictures. We will collect the triangles for a *PASCAL*-Poster.

Here are three *DERIVE*-plots showing the even numbers, the numbers which are divisible by nine and the remainders after division by 5.
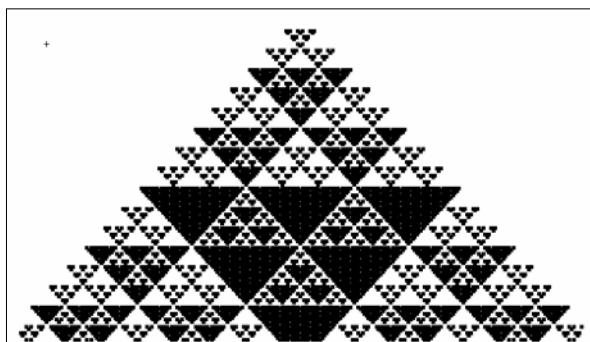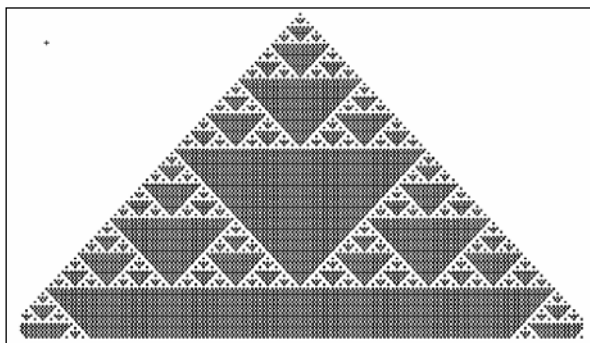
$$
\begin{bmatrix}
[1] \\
[1,1] \\
[1,2,1] \\
[1,3,3,1] \\
[1,4,1,4,1] \\
[1,0,0,0,0,1] \\
[1,1,0,0,0,1,1] \\
[1,2,1,0,0,1,2,1] \\
[1,3,3,1,0,1,3,3,1] \\
[1,4,1,4,1,1,4,1,4,1] \\
[1,0,0,0,0,2,0,0,0,0,1] \\
[1,1,0,0,0,2,2,0,0,0,1,1] \\
[1,2,1,0,0,2,4,2,0,0,1,2,1] \\
[1,3,3,1,0,2,1,1,2,0,1,3,3,1] \\
[1,4,1,4,1,2,3,2,3,2,1,4,1,4,1]
\end{bmatrix}
$$

The program code for pas(n,a,r):

```
pas(n,a,r)
Prgm
ClrDraw
FnOff :PlotsOff
setGraph("Axes","Off")
setGraph("Grid","Off")
Local i,j,z,pt
Define pt(x,y)=Prgm
PxlOn x,y:PxlOn x,y:PxlOn x+1,y
PxlOn x,y-1:PxlOn x+1,y-1
EndPrgm
For i,Ø,n
  For j,Ø,i
    nCr(i,j)→z
    If mod(z,a)=r Then
    pt(1+2*i,117-2*i+4*j)
    Else
    PxlOn 1+2*i,116-2*i+4*j
    EndIf
EndFor
EndFor
EndPrgm
```

Two explanations:

nCr(i,j) generates the *binomial coefficients* using a rule which you will learn later in probability theory.

Function mod(z,a), *Modulo Function*, returns the integer remainder after division of the integer z by the integer a.

Find the following using mental computation then using your device:

mod(87,9) = ...

mod(15,3) = ...

mod(113,7) = ...

mod(2ØØ,13) = ...

Which line of the program above would you will have to change to change pas() to invpas(), to return the *inverse* figures?

Assessment example (C3)

Given are some terms of a binomial expansion. Complete the unknown terms

a) $x^4 - 16x^3 + 96x^2 -$ ......

b) $a^6b^3 + 6a^4b^5$ .... = (..........................)^3

c) .... $- 180x^3z$ .... = (..........................)^2 Give more than one solution!

d) $729x^{12} - 5832x^{10}y + 19440x^8y^2$ .... =

On the next page we can see the results of the efforts to create a fine poster.

Our PASCAL Gallery