NUMB3RS Activity: Traveling on Good Circles
Episode: “Spree, Part I”

Topic: Geodesics on a sphere **Grade Level:** 11 - 12
Objective: Find the shortest great circle distance between two cities.
Time: 30 - 40 minutes
Materials: a scientific or graphing calculator

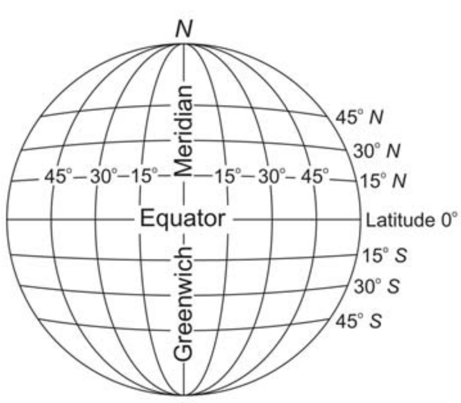
Introduction

In “Spree, Part I,” Don and Edgerton have plotted a precise map of a crime spree carried out by two lovers. The spree ranges from Austin, TX, to San Bernardino, CA, and goes (in order) through the cities Mesquite, TX; Oklahoma City, OK; McPherson, KS; Colby, KS; Ft. Collins, CO; Salt Lake City, UT; Milford, UT; and Las Vegas, NV. Charlie thinks the criminals visited another city along the way, but did not commit a crime there. He identified this “mystery city” by finding the distances between other pairs of cities on the spree using a formula involving the latitude and longitude of each city. In this activity, we investigate such a formula.

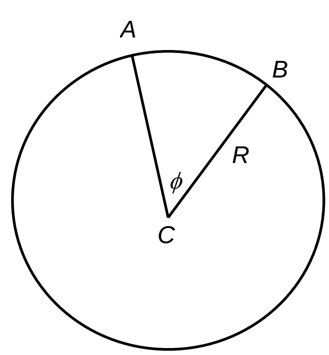
Discuss with Students

Ask students if any of them have ever taken a long flight to Europe or Asia where the pilot indicated they were following a “great circle route.” The shortest route from Los Angeles to London, for example, goes northeast through central Canada, turns east across the southern tip of Greenland, and arrives in London from the northwest. Great circles on a sphere, those formed by the intersection of the sphere and a plane containing its center, are **geodesics** – curves of shortest length between two points on a sphere. To make this statement believable, you can consider one or more of these demonstrations:

- Take a strip of paper and place it on a sphere. It lays flat if it lies on a great circle.
- Take a rubber band, Put it on the sphere. If it "stays in place” and does not “fly off” the sphere, it is a great circle.
- Put a drop of water on the top of a sphere. The path it follows is a great circle.
- Roll a ball on a straight chalk line (or straight on a freshly painted floor!). The chalk (or paint) will mark the line of contact on the sphere, and it will form a great circle.



[Source: Adapted from Geometry, Roger Fenn, Spring Verlag, 2001, ISBN 1-85233-58-9, p. 261]



To compute great circle distances, you will need to review the meaning of latitude (the angular measure of the earth's central angle north or south from the equator), longitude (the angular measure of the central angle east or west of the prime meridian), and the arc length formula $S = R\phi$. The equator has latitude 0° and the prime meridian (through Greenwich, England) has longitude 0°. Because we will be using trigonometric functions, instead of using north (N) and south (S), latitudes north of the equator will have positive values while latitudes south of the equator will have negative values. Similarly, points east of the prime meridian will have positive longitudes while points west of the prime meridian will have negative longitudes. You might want to bring in a Styrofoam ball and cut out a central angle so that students can "see" the latitude and longitude and the arcs involved.

Also used in the derivation of the great circle distance formula are the Cartesian Distance Formula for two points in space and the Law of Cosines. This derivation is posed as a question in the Extensions.

Student Page Answers:
1. $c = 0$, $-180 \le d \le 180$ **2.** $-90 \le e \le 90$, $f = 0$ **3.** about 7,733 miles **4.** about 621 miles
5a. arc length = $R\psi$ but $R = 1$ **5b.** $PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 2 - 2\cos\psi$ so that $\cos\psi = x_1x_2 + y_1y_2 + z_1z_2$ since $x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2 = 1$ **6.** 2.732 **7.** Answers vary but will use the definitions of sine and cosine applied to the triangle with hypotenuse *OP*. **8.** about 621 miles
9a. about 613 miles **9b.** about 612 miles **9c.** compared to the circumference of the Earth, the two cities are 'close together' **10a.** about 9,366 miles **10b.** about 7,327 miles

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Name: _________________________________ Date: _______________

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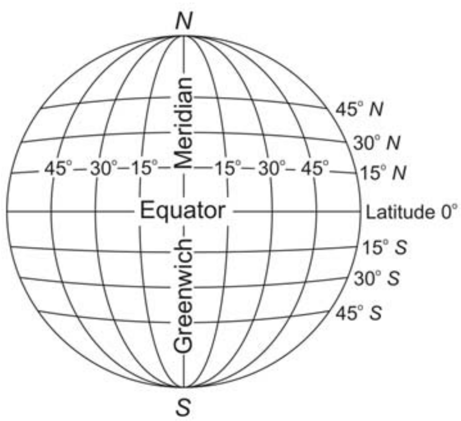

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1. If a point on the equator has (latitude, longitude) = (c°, d°), what values are possible for c and d?

2. If a point on the prime meridian has (latitude, longitude) = (e°, f°), what values are possible for e and f?

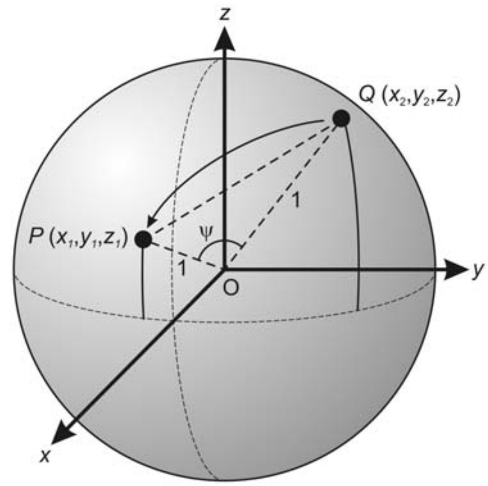
3. Find the great circle distance between Quito, Ecuador (0°, 79°W) and Kampala, Uganda (0°, 33° E). [Hint: First, find the circumference of the Earth. Then find the angle that is formed at the center of the great circle (equator) by the radii to these two cities.]

4. Find the great circle distance between St. Louis (39°N, 90°W) and New Orleans (30°N, 90°W).

To develop a formula for finding the great circle distance between any two cities on the Earth, consider the formulas on a sphere of radius 1. The corresponding great circle distance on the Earth, then, is found by multiplying our results by 3,956 miles – the radius of the Earth.

Cartesian Distance

5. Consider a sphere of radius 1.



a. The great circle distance PQ equals ψ ($m\angle POQ$) measured in radians. Why?

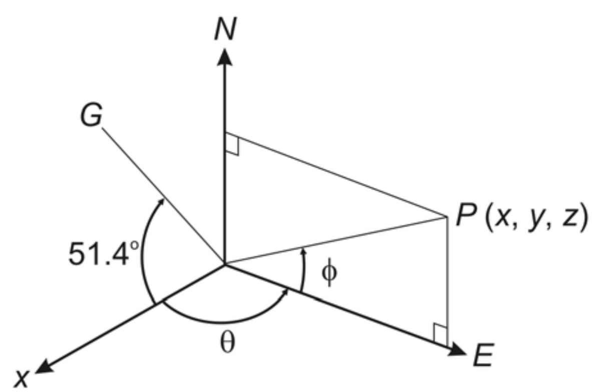
We can find ψ by finding cos ψ using the Law of Cosines in $\triangle OPQ$. Thus, $PQ^2 = 1^2 + 1^2 - 2\cos\psi$.

b. Show cos $\psi = x_1x_2 + y_1y_2 + z_1z_2$ using the Cartesian Distance formula for two points in space.

6. Find the perimeter of the spherical triangle whose vertices are $\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)$, $\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, and $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$.

Distance Using Latitude and Longitude

7. This figure represents part of a unit sphere. Let G correspond to Greenwich. Then arc *NGX* would be the "prime meridian" and θ would be the **longitude** of point *P*. Similarly if arc *XE* is part of the equator, then ϕ would be the **latitude** of point *P*. Then we can express *x*, *y*, and *z* in terms of θ, and ϕ as follows:



$x = \cos\phi \cos\theta$
$y = \cos\phi \sin\theta$
$z = \sin\phi$

Explain in your own words why these relationships are valid.

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To express cos ψ in terms of ϕ and θ, substitute these relationships into the formula cos $\psi = x_1x_2 + y_1y_2 + z_1z_2$ obtained above. The great circle distance between two points *P* and *Q* with latitude and longitude ϕ_1, θ_1 and ϕ_2, θ_2 is ψ (in radians), where

$$\cos\psi = \cos\phi_1 \cos\phi_2 \cos(\theta_1 - \theta_2) + \sin\phi_1 \sin\phi_2 .$$

The distance, in miles, between two points on the Earth with latitude and longitude ϕ_1, θ_1 and ϕ_2, θ_2 then is $3{,}956\psi \cdot \frac{\pi}{180}$, where ψ is found using the inverse cosine and is measured in degrees. (Note that if you find ψ in radians, then the distance is simply 3956ψ).

8. Use this formula to verify the great circle distance between St. Louis and New Orleans found in Question 4.

9. a. Find the great circle distance (in miles) between Chicago (42°N, 88°W) and Washington, DC (39°N, 77°W).

b. Suppose you dug a straight-line tunnel between Chicago and Washington, DC. How many miles long would this tunnel be?

c. Why are your answers to parts a and b approximately equal?

10. a. Find the great circle distance (in miles) between Chicago (42°N, 88°W) and Singapore (1°N, 104°E).

b. Suppose you dug a straight-line tunnel between Chicago and Singapore. How many miles long would this tunnel be?

The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

For the Student

1. Find the great circle distance between Austin, TX, and San Bernardino, CA (the starting and ending cities of the crime spree in this episode). (Hint: you will first need to find the latitude and the longitude of each city.)

2. There are many “great circle route” calculators on the Internet. Some samples are listed below. Use one of these Web sites to find the great circle distances between several pairs of cities that you would like to visit.

- How Far Is It: **http://www.indo.com/distance**
- Great Circle Mapper: **http://gc.kls2.com**
- Great Circle Navigation Calculator: **http://www.csgnetwork.com/marinegrcircalc.html**

3. If you compute ‘short’ great circle distances with this formula on your calculator, the results might not be very accurate. Why?

4. Many “navigators” measure distances in nautical miles instead of the more familiar statute miles. What is the difference between a statute mile and a nautical mile? How could you modify the formula for great circle distance to express the results in nautical miles?

5. Derive the formula $\cos \psi = \cos \phi_1 \cos \phi_2 \cos(\theta_1 - \theta_2) + \sin \phi_1 \sin \phi_2$.

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