

Activity 2

Objectives

- Gain an understanding of the Law of Large Numbers
- Formulate a hypothesis on the fairness of a model

Law of Large Numbers: A Weighty Decision

Introduction

In the previous activity, you found that it might take thousands of rolls of a fair die before experimental probability comes close to matching theoretical probability. The bars of a graph displaying the results of a fair die will most likely be of varying heights, but may begin to even out as the number of rolls increases. Imagine that you have an unfair model—either a weighted die or a spinner with unequal sections.

Problem

How can you determine if the difference in the height of the bars is due to the expected variation in experimental results or the unfairness of the model? How much evidence (how many rolls or spins) would you need to make a determination on the fairness of a spinner or die?

Exploration

Getting Started

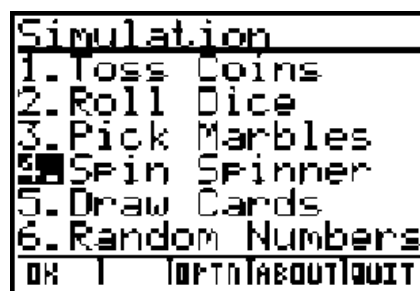
Looks are deceiving. The spinner on the teacher's graphing calculator is not fair. One and only one section is either slightly larger or slightly smaller than the others. The remaining sections are exactly equal in size. Watch as it spins 200 times. Can you tell which section is a different size than the other sections?



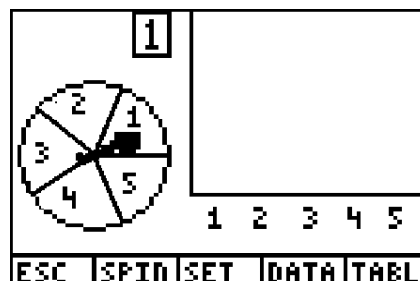
Respond to # 1 on the Student Worksheet.

Weighting a Spinner

1. Open the Probability Simulation application and select **Spin Spinner** from the **Simulation** menu.



2. Select **SET**, and then change the settings as shown.
3. Select **OK**.



All five sections have equal areas. Remember that with a fair spinner, the pointer has an equally likely chance of stopping in any section. Because there are five equal sections, the probability of the pointer stopping in any section is 1 out of 5, which can be represented as $\frac{1}{5}$, 0.2, or 20%. However, if one section is weighted more than the others, the pointer is more likely to stop on that section.

With the Probability Simulation application, you can set the weight or size of each section.

4. Select **SET** to return to the **Settings** menu.
5. Select **ADV**, change the weight of Sctn 1 to **2**, and then press **ENTER**.
The probabilities update automatically.

Sctn	Weight	Prob
1	2	.3333
2	1	.1667
3	1	.1667
4	1	.1667
5	1	.1667

ESC OK

6. Select **OK** twice to view the spinner.
Notice that the spinner still has five sections, but section 1 is twice as large as each of the others.



Respond to # 2 – 8 on the Student Worksheet.

7. Select **SET** to return to the **Settings** menu.
8. Select **ADV**, change the weight of Sctn 2 to **2**, and then press **ENTER**.
Notice how this changed the probabilities of each section. Continue to change the weight of the other sections to **2**. Notice what happens to the probabilities as you change each section. Look for patterns. Predict the probabilities before you change the weight each time.
9. Change all the weights to **3** following the same section by section procedure.



Respond to # 9 – 10 on the Student Worksheet.

10. Select **OK** to return to the **Settings** menu.
11. Select **ADV** and change the weights as shown.

Sctn	Weight	Prob
1	4	.2
2	2	.1
3	3	.3
4	1.5	.15
5	2.5	.25

ESC OK

12. Select **OK** to return to the **Settings** menu, and then change the settings as shown.
13. Select **OK**.

Settings	
Trial Set:	100
Sections:	5
Graph:	Area Prob
StoTbl:	No All 50
ClearTbl:	Yes No
Update:	20 50 End

ESC ADV OK



Respond to # 11 – 14 on the Student Worksheet.

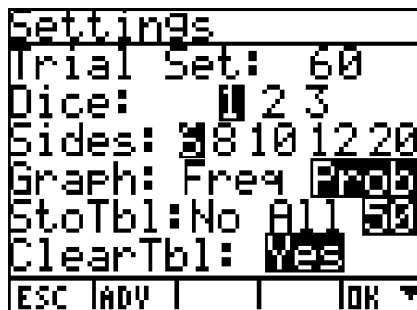
Weighting a Die

Unlike an unfair spinner, a weighted die does not provide visual clues that it is unfair. In this part of the Exploration, you determine if a die is fair or unfair.

1. Open the Probability Simulation application and select **Roll Dice** from the **Simulation** menu.



2. Select **SET**, and then change the settings as shown.

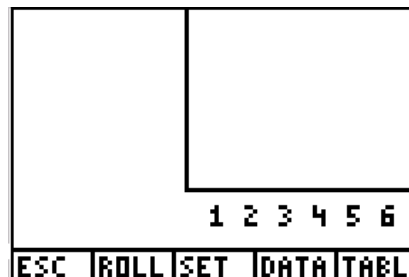


3. Select **ADV**, and then change the weights for each side as shown.

Sctn	Weight	Prob
1	1	.1
2	2	.2
3	1	.1
4	2	.2
5	3	.3
6	0	0

Navigation options at the bottom: ESC, |, |, |, OK.

4. Select **OK** twice.



Respond to # 15 – 17 on the Student Worksheet.

5. Select **ROLL**. The Probability Simulation App rolls a die 60 times while displaying the results in a dynamic bar graph. Press \leftarrow or \rightarrow to scroll from one bar to the next to view the frequency of occurrence of each side.



Complete the *first* graph only in # 18 on the Student Worksheet.

- Clear the data from the first trial of 60 rolls. To clear data, select **SET**, scroll down to **ClearTbl**, and then highlight **Yes**. Press **ENTER** and then select **OK**. The graph is blank if you cleared the table successfully.
- Run the simulation two more times.

Note: If you did not clear the previous trial of 60 rolls and you already started rolling again, press **ON** to interrupt the simulation, then clear the table as described.



Complete the second and third graphs in # 18 on the Student Worksheet. Next, complete # 19 – 22 on the Student Worksheet.

- Run three more trials of 60 rolls *without* clearing the table. This will give you data for 240 rolls.

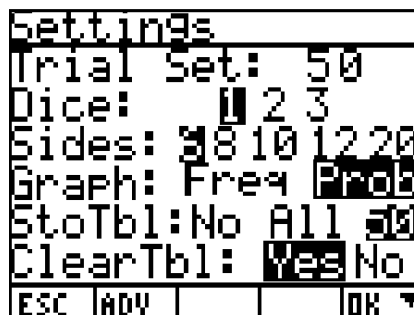


Respond to # 23 – 25 on the Student Worksheet.

Extension

For this activity, you will work with a partner to play the game, “A Weighty Decision.” The objective of the game is to determine if a die is weighted and, if so, to determine the weights assigned to each side.

- Each player secretly sets the weights on his or her graphing calculator. Select **SET**, and change the settings as shown.



- Select **ADV** and set the weights for each side. The weights must total 20. Record the weights on a piece of paper so you can give feedback to your opponent. *Do not let your opponent see the weights you select!*
- Select **OK** twice to return to the simulation screen and then exchange graphing calculators with your partner.
- Select **ROLL** and then record the experimental probabilities (to the nearest hundredth) on the game card on the Student Worksheet.
- Predict the weighting of the die or pass. If you make a prediction, record it on the game sheet and then share it with your opponent. Your opponent says only that the prediction is correct or incorrect. If the prediction is incorrect, roll the die again and then make another prediction.

Note: Do not clear the table after each trial. Allow the rolls to accumulate.

6. The game is over when both players have guessed correctly the weighting of the die.

Scoring:

Points are recorded for each round.

Action	Points Awarded	Points Deducted
Pass		1
Correct guess	10	
Incorrect guess		Five points plus the difference between each guessed weight and the actual weight

Example: Player A

	Predicted Weighting	Actual Weighting		Points
Round 1	3, 3, 5, 1, 5, 3	3, 3, 4, 2, 5, 3	7 pts deducted	-7
Round 2	pass		1 pt deducted	-1
Round 3	3, 3, 4, 2, 5, 3	3, 3, 4, 2, 5, 3	10 pts awarded	10
			Total Points	2

Student Worksheet

Name _____

Date _____

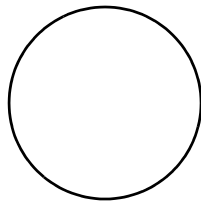
Law of Large Numbers: A Weighty Decision

1. After each set of spins, predict which section of the spinner is a different size. Circle if it is larger (L) or smaller (S) than the other sections. Explain your thinking.

Spins	Predictions	Reason
200	Section # L or S	
400	Section # L or S	
600	Section # L or S	
800	Section # L or S	
1,000	Section # L or S	
1,200	Section # L or S	

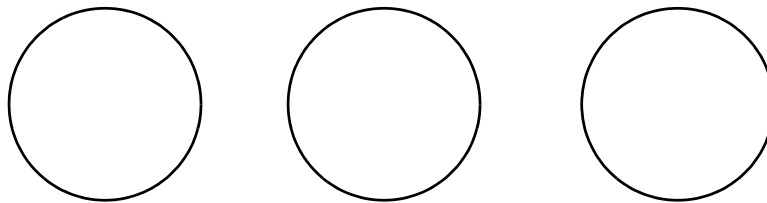
2. What is the probability of the pointer stopping in each section when all the weights are 1? What is the sum of the probabilities of all five sections?
3. What is the probability of the pointer stopping in the first section if you change the weight of the first section to 2?
4. Did doubling the weight of the first section double its probability?
5. How does doubling the weight of only the first section affect the probabilities of all of the sections? What is the sum of the probabilities of all five sections? Explain.

6. Draw a spinner with six equal sections. Label the sections using the numerals 1 – 5 so that the probability of occurrence for landing on each *number* would be the same as the five-section weighted spinner on your graphing calculator.



7. Explain why the spinner you drew in # 6 and the spinner shown on your graphing calculator have the same probability of occurrence for each number.

8. Design three different spinners, each with a different number of sections and labeled with the numerals 1 – 4, such that the probability of spinning 1 is 25%, 2 is 37.5%, 3 is 12.5%, and 4 is 25%.



9. What patterns do you notice as you change the weight of each section to 2? Do you notice similar patterns as you change the weight of each section to 3?

10. Does it surprise you that when all the weights are 2, the probability of occurrence is the same as when all the weights are 1? How does this compare to having all sections weighted with 3? Weighted with 4? Explain.

11. Your five-section spinner is weighted as given in the chart below. Describe a mathematical method the Probability Simulation App might use to determine the theoretical probability of each outcome.

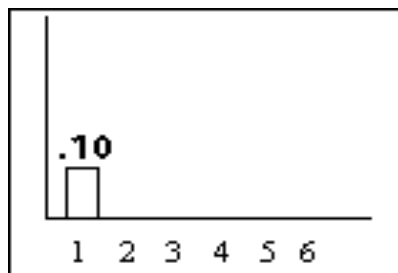
Fill in the Theoretical Probability column in the table below.

Section	Weight	Theoretical Probability	Expected after 100 Spins	Results after 100 Spins	Expected after 400 Spins	Results after 400 Spins
1	4					
2	2					
3	6					
4	3					
5	5					
Total						

12. Out of 100 spins, how many times would you expect the pointer to stop on each number? Out of 400 spins? Record your expected outcomes (theoretical frequencies) in the table above.
13. Record the experimental results of 100 spins and 400 spins in the table above.
14. How closely do your experimental results match the expected (theoretical) results? Give reasons for any discrepancies.
15. Sketch a bar graph to represent the theoretical probability distribution of a die that is weighted as shown below. On top of each bar write the theoretical probability of occurrence rounded to the nearest hundredth.

Weights:

- Side 1 weighted 1
- Side 2 weighted 2
- Side 3 weighted 1
- Side 4 weighted 3
- Side 5 weighted 3
- Side 6 weighted 0



16. What is the sum of the probability of occurrence of all six sides?

17. What does it mean to have a probability of 0?

18. Sketch the experimental distribution for three separate 60-roll trials. Write the probability of occurrence rounded to the nearest hundredth above each bar.



Experimental Distribution #1
60 rolls



Experimental Distribution #2
60 rolls



Experimental Distribution #3
60 rolls

19. What is the sum of the experimental probability of occurrence for each of the distributions?

20. How closely does each experimental distribution match the theoretical probability distribution you sketched in # 15?

- 21.** What do you notice about the height of the bars above number 6? Does this agree with the theoretical probability distribution? Explain.
- 22.** How similar are the experimental distributions to each other? How similar are they to those of your classmates?
- 23.** Sketch the experimental distribution for 240 rolls. Write the probability of occurrence rounded to the nearest hundredth above each bar. How closely does your experimental distribution match the theoretical distribution in #15? How similar is your experimental distribution to your classmates' graphs?



Experimental Distribution
240 rolls

- 24.** You have been investigating the Law of Large Numbers. Explain in your own words what you understand about the Law of Large Numbers.
- 25.** Because of the Law of Large Numbers, casino owners and insurance companies are guaranteed to make a profit while their customers are not. Explain this statement.

Extension

Game Cards for A Weighty Decision

Round	Total Number of Rolls	Experimental Probability of Occurrence						Prediction or Pass?	Predicted Weighting						Points	
		1	2	3	4	5	6		1	2	3	4	5	6		
1																
2																
3																
4																
5																
6																
7																
8																
9																
10																
Total Points																

Round	Total Number of Rolls	Experimental Probability of Occurrence						Prediction or Pass?	Predicted Weighting						Points	
		1	2	3	4	5	6		1	2	3	4	5	6		
1																
2																
3																
4																
5																
6																
7																
8																
9																
10																
Total Points																

Teacher Notes



Activity 2

Law of Large Numbers: A Weighty Decision

Objectives

- Gain an understanding of the Law of Large Numbers
- Formulate an hypothesis on the fairness of a model

Materials

- TI-84 Plus/TI-83 Plus
- ViewScreen™ panel or TI-Presenter™ video adapter

Teaching Time

- 90 minutes

In Activity 1, students explored questions of theoretical and experimental probability with a fair die. In Activity 2, students expand their understanding of probability by looking at unfair models, such as an unfair spinner and a weighted die. They predict the weighting of an unfair model by analyzing experimental data and distributions.

Preparation

The Exploration launches with a whole-class demonstration of an unfair four-section spinner. Run the simulation of the unfair spinner. (Use a View Screen™ panel or TI-Presenter™ video adapter so that students can view the demonstration.) After every 200 spins, students should record the relative frequency of each section and make a prediction as to which section is larger or smaller than the others. Encourage students to explain their thinking.

In the rest of the Exploration, students examine experimental results of unfair models. Students set the weight for the spinner and have a visual of the unfair model. As they run simulations, they see the dynamic graph of the results. After each trial, students are asked to look for patterns in the results. They are directed to compare the experimental results with the theoretical probability.

Graphing Calculator Set-up

1. Open the Probability Simulation application and select **OPTN**.
2. Set the seed to **2** and then select **OK** to return to the **Simulation** menu.
3. Select **Spin Spinner**.
4. Select **SET** to display the settings menu.
5. Change Trial Set to **200** and Graph to **Prob**.
6. Select **ADV** to change the weights for sections 1, 2, and 3 to **100**, and section 4 to **95**.

7. Select **OK** twice to see the spinner. Even though section 4 is weighted less than the other sections, it appears to be the same size. The screen resolution of the graphing calculator does not reflect the slight difference in weights.
8. Select **SPIN**. The spinner spins 200 times and the bar graph dynamically updates itself after each spin.
9. After each set of 200 spins, press \leftarrow or \rightarrow to display the relative frequency of each outcome. Record this information on a class chart for students to view. (See Transparency for Class Chart at the end of this section.) At the same time, students should complete #1 on the Student Worksheet.

Note: Step # 2 advises setting the random number generator seed to 2. This means that you will spin the same sequence of 1,200 spins that the authors did. This particular set of 1,200 spins supports a variety of student predictions assuring the opportunity for a lively discussion. If you want this same set of 1,200 rolls after you have explored the activity as a teacher, you must reset the seed to 2 immediately prior to starting the demo with the class. Any extra spins will alter the outcome. Of course, you may find other seed numbers that offer more exciting outcomes after each set of 200 rolls.

10. After each set of 200 spins, poll the students asking which section is different. Record the poll results on the class chart. Ask students to justify their predictions.
11. After the class discussion for 1200 spins, show the students the Advanced Setting screen to reveal the weights assigned to each section.

Weighting a Spinner

This part of the exploration sets the stage for students to construct a mathematical procedure to calculate theoretical probability from the given weights of a biased model. Proportional reasoning is emphasized. Using the spinner simulation allows students to visually compare the relative size of each section. If one section increases in size, the others must decrease. The probabilities must change accordingly.

Weighting a Die

In this part of the exploration, students look at experimental results presented in the form of bar graph distributions and determine whether a model is fair or unfair. They create an unfair model (a weighted die) and run multiple sets of 60 trials on the graphing calculator. They see the dynamic graphing of the experimental results and begin to discern the difference between normal variance in experimental results and results of a biased experiment. After 240 trials, the shape of the distribution more consistently resembles the shape of its theoretical distribution. However, the experimental probabilities may not accurately approximate the theoretical probabilities.

As in Activity 1, the class may want to pool the results of each student's 240 rolls to see how closely the results after thousands of rolls approximate the theoretical probabilities.

Extension

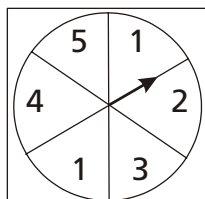
The game "A Weighty Decision" motivates students to apply their understanding of the Law of Large Numbers to predict the theoretical distribution when the weights assigned to each side of a die are unknown.

Answers to Student Worksheet

1. Answers will vary. For a seed of 2, the relative frequencies are as follows:

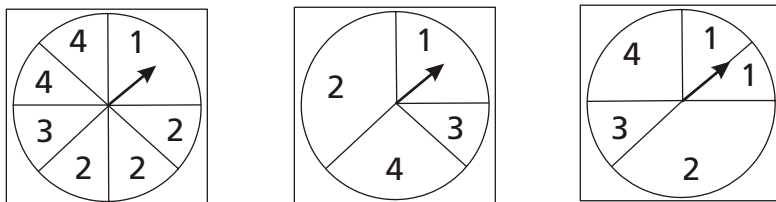
Relative Frequencies						
Outcome	200 Rolls	400 Rolls	600 Rolls	800 Rolls	1000 Rolls	1200 Rolls
1	0.285	0.2775	0.2666	0.2525	0.252	0.2483
2	0.28	0.27	0.26	0.2625	0.259	0.2608
3	0.195	0.2225	0.2433	0.245	0.255	0.2625
4	0.24	0.23	0.23	0.24	0.234	0.2283

2. 0.2, 1
3. 0.3333
4. No, it did not double.
5. The probability of the pointer stopping in the first section increased from 0.2 to 0.3333 but the probabilities for the other sections decreased from 0.2 to 0.16667. The sum of the probabilities is 1.0001 (slightly more than 1 due to rounding). The total sum of the probabilities for all outcomes must equal 1. In order for the sum of the probabilities to remain 1, when the probability of the pointer stopping in the first section increases, the probability of the pointer stopping in the other sections must decrease.
6. Answers will vary; two sections must be labeled 1. One possibility is shown.



7. In both circles, the area occupied by sections labeled with 1 (one section or two) is the same and equals $\frac{1}{3}$ of the circle.

8. Answers may vary; one possibility is shown.



9. Answers will vary. The sum of the probabilities of all five outcomes is 1. As the weight of each changed section increases, the weights of the other sections decrease. Sections with equal weights have equal probabilities. When all the weights are the same, the probability of each of the five sections equals 0.2.

10. Answer will vary. Students should recognize that if the weights for each section are the same, whatever value they may be, the probability of occurrence will be the same (0.2).

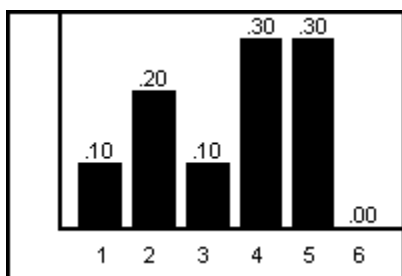
11. The probability of a section equals the weight of a section divided by the sum of all the weights.

Section	Weight	Probability	Expected after 100 Spins	Results after 100 Spins	Expected after 400 Spins	Results after 400 Spins
1	4	0.2	20		80	
2	2	0.1	10		40	
3	6	0.3	30		120	
4	3	0.15	15		60	
5	5	0.25	25		100	
Total	20	1.00	100		400	

12. See chart above.

13. Answers will vary.

14. Answers will vary. Possible reasons: 400 spins are not enough for experimental results to match the theoretical frequencies; it is more appropriate to compare ratios than it is to compare frequencies.



15.

16. 1

- 17.** A probability of 0 means that the outcome is impossible; a 6 will never be rolled.
- 18.** Students' graphs will vary.
- 19.** The sum of the probabilities for each distribution is 1, except for slight variation due to round off error.
- 20.** Answers will vary.
- 21.** There should be no bar above 6 since it has a probability of 0. When the theoretical probability is zero, the experimental frequency of occurrence must also be zero. Realize that an experimental probability of zero does not imply a theoretical probability of zero.
- 22.** Answers will vary. Emphasize that with only 60 rolls, the shape of many distributions may not even resemble the shape of the theoretical distribution.
- 23.** Answers will vary. Most distributions will be similar in shape to the theoretical distribution; however, the experimental probabilities may not accurately approximate the theoretical probabilities.
- 24.** Answers will vary. Students should realize that the greater the number of trials, the more likely it is that the experimental probability will approximate the theoretical probability.
- 25.** The odds are in favor of both casino owners and insurance companies. The thousands of bets being placed each night represent the experimental results. The Law of Large Numbers assures the casino owner that the overall results will be quite close to the house's favored theoretical advantage. On the other hand, a single gambler will place only several hundred bets in a night. This is not nearly enough to guarantee a winning evening.

An insurance company's actuarial tables favor the odds that the company will make a profit. The company must have thousands of customers for the Law of Large Numbers to assure this profit. A single customer, on the other hand, has very few opportunities to make a claim. Thus sometimes his total insurance premiums paid out may be more than the total amount of the claim, in which case he has a net loss. However, it might be that the amount of one claim largely exceeds the premiums an individual paid.

Transparency for Class Chart

Spins		Outcomes			
		1	2	3	4
200	RF				
	Poll				
400	RF				
	Poll				
600	RF				
	Poll				
800	RF				
	Poll				
1000	RF				
	Poll				
1200	RF				
	Poll				

RF - Relative Frequency